Toward a resolution of the black hole information paradox: the quasi-analytic transition between self-gravitating strings and black holes

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String theory has provided a powerful framework for uniting quantum gravity and semiclassical black hole thermodynamics. In this paper, we begin with the origins of the black hole information paradox in the semiclassical regime, analyzing Hawking and Bekenstein's calculation of the entropy of a black hole via quantum field theoretic techniques. We then trace the development of string theory, first through Kaluza-Klein mechanism and progressing through the computation of scattering amplitudes, eventually reaching the Horowitz-Polchinski correspondence. The final goal of this outline is to reach an understanding of the critical physics between the string and black hole scales, interpolating between self-gravitating string solutions and quantum-corrected black holes. A key result is the derivation of the Bekenstein-Hawking entropy formula from the stretched horizon and string microstate counting; we show an agreement in entropy to order unity between string states and black hole states. Particularly, we review a recent work by Chen, Maldacena, and Witten, which derives an explicit, quasi-analytic solution in a supersymmetric regime for 3 noncompact dimensions. This solution matches the black hole entropy to leading order with a cubic correction term and is valid at scales up to small deviations from the Hagedorn radius.

1. CLASSICAL BLACK HOLES

Originally thought to be an unreachable limit of Einstein's field equations, the Schwarzschild solution describes a non-rotating, uncharged, static black hole of a given mass M [24]. Further solutions for black holes with charge (Reissner-Nördstrom) and/or angular momentum (Kerr-Newman) were later found, however much of the interesting physics relevant for this paper can be understood through the Schwarzschild solution:[1]

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{1.1}$$

$$d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2 \tag{1.2}$$

This solution has a few important features. Besides the typical event horizon through which no light can return, there is also a point at $r = 0$ where the curvature of spacetime diverges. Though originally thought to be a consequence of the formalism, singularities were proved to exist in the 1960s by Rodger Penrose [23]. This eventually led to the Hawking-Penrose Singularity theorems, which defined the specific mass-energy conditions that necessitated the evolution of a singularity.

1.1. *Singularities and the no-hair theorem*

Another particularly relevant feature of semiclassical black hole solutions is the dimension of their configuration space. According to even the most general metric, the Kerr-Newman metric, the only parameters required to specify a black hole are its mass (M), charge (Q), and angular momentum (J). This was coined by John Wheeler as the "No-hair theorem" [4], representing the apparent simplicity and uniqueness of the black hole structure. This theorem implies that any two electrically charged, rotating black holes which share the same parameters are indistinguishable from outside the event horizon, regardless of their internal constitution or formation processes. The term "No-hair" refers to the idea that black holes have no other observable features, or "hair", beyond these three classical parameters. This simplicity arises from the extreme gravitational forces near the black hole's event horizon, which is understood semiclassically as a smooth, causally-unidirectional boundary.

2. QUANTUM ELECTRODYNAMICS AND QUANTUM FIELDS

Concurrent to the development of classical black hole theory, General Relativity was also adapted to new quantum regimes. This came first in the form of Quantum Electrodynamics, a theory first developed to account for general relativity in Maxwell's Electromagnetic field equations, as well as in other conundrums, such as the Photoelectric effect, which left first-generation quantum mechanics clueless.

This theory had been pioneered by Paul Dirac, who, through incredible mathematical analysis of electromagnetic structure in the context of quantum mechanics, developed via canonical quantization a relativistic electromagnetic field equation; now known as the Dirac equation:

$$(i\gamma^\mu \partial_\mu - m)\psi = 0 \tag{2.1}$$

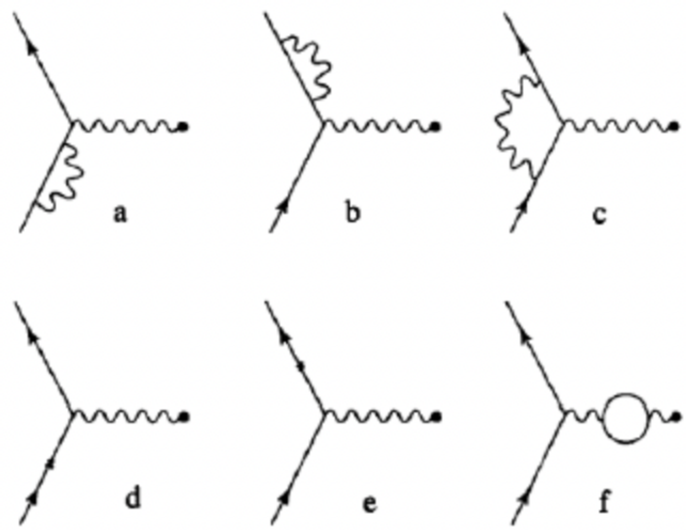
Where γ^μ represent a commutation algebra compatible with the Minkowski (background) metric in General relativity [10].[2] This theory began a decades-long development of quantizing fields under the regime of General Relativity; characterizing interactions by perturbations. This is, of course, the central idea of Quantum Field Theory. Further developments of this theory by Oppenheimer, Schwinger, and Feynman developed this picture of perturbative interactions into its modern "S-Matrix" form. This was a necessary physical consequence of the quantized understanding of matter: interactions could happen at any perturbative order, and thus to understand the evolution of a quantum state required a summation over an infinite perturbative series, with each successive order's magnitude determined by the fine structure constant $\alpha \approx 1/137$.

[1] For the entirety of this paper, we employ natural units, setting $c = \hbar = 1$

[2] Books could be dedicated to this single equation alone, but for the sake of volume we will not elaborate further here.

2.1. *Perturbative QFT and renormalization*

More importantly, however, was the problem of divergences. Dirac's original insight led quickly to problems with infinities on both ends of the energy spectrum; Quantum field interactions quickly ran into both Ultraviolet (UV) and Infrared (IR) diverges, and the mathematicians had resorted to guesswork based on experimental evidence. However, by the 1950s, a method of dealing with these infinities known as renormalization came to the forefront of theoretical physics. This approach involved anchoring diverges to experimental values and adjusting the physics at divergent length scales to remove divergences. The associated physics is recognized as Feynman diagrams, which quickly became a staple calculating scattering amplitudes in QFT.



A collection of higher-order Fenyman diagrams representing corrections to the Lamb Shift[3]

Through renormalization excited physicists at the time, it amounted to a ptolemization of the theory. Quantum Field Theory quickly ran into more problems: some interactions, such as the Weak interaction (described to mediate Hadronic interactions), were non-renormalizable. This is not unexpected; the whole scheme was to deal with a break-down of a well-defined theory through repeated modification and ultra-specification of the rules; something nature is not known for doing. Additionally, QFT was aimless at describing gravity, as gravity had no quantization procedure which was consistent with evidence.

2.2. *Yang-Mills theory*

Though Quantum Field Theory had many issues, there was a mathematical formalism developed alongside it which attempted to generalize the field interactions beyond the simplistic field description. Yang-Mills Theory, as it came to be called, was generalized from the plethora of QFTs developed in various domains of physics to deal with quantization [32]. Yang-Mills theory is a non-abelian gauge theory which describes a general scheme for quantized interactions. It is based in an action functional derived from $SU(n)$ symmetry, and puts rigid constraints on the semiclassical limit of any "theory of everything".

3. SEMICLASSICAL BLACK HOLE THERMODYNAMICS

Black holes were not immune to this revolution of quantization. In 1973, Hawking and Bardeen put forth the "Four laws of Black Hole thermodynamics" which characterized Black holes as statistical systems with entropy and temperature [3]. While some laws are canonical, some remain contentious in the context of modern physics, as they heavily draw on an analogy between classical thermodynamic systems and black holes. The third law in particular

[3] The Lamb shift was a divergence in the vacuum energy of a quantum field through the creation-annihilation of virtual photons. This was one of many issues solved via renormalization

has been viewed by modern physics with much scrutiny [29]. Particularly relevant to us, however, is the second law of black hole thermodynamics. Stated in its original form, it claims the following:

The second law of black hole thermodynamics: The area A of the event horizon of each black hole does not decrease with time, i.e. $\delta A \geq 0$. If two black holes coalesce, the area of the final horizon is greater than the sum of the areas of the initial horizons, i.e. $A_3 > A_1 + A_2$.

3.1. *Hawking radiation and the Unruh effect*

This analogy led one of Feynman's graduate students, Jacob Bekenstein, to extend this thermodynamic. Together with Hawking, they formulated a definite entropy and temperature for a black hole [5]. They are presented here in their original form:

$$T_{BH} = \frac{1}{8\pi M} \tag{3.1}$$

$$S_{BH} = \frac{k_B A c^3}{4G\hbar} \tag{3.2}$$

This now-famous Bekenstein-Hawking entropy implies something unusual about the entropy of a black hole: it is proportional not to its volume, but its surface. This implied that, if unitarity were to be preserved, the infalling information lost beyond the horizon of a black hole must be proportional to its surface.[4]

There was another important discovery made around the same time as the above revelation: black holes, being thermodynamic objects, also emit black-body radiation, known as Hawking radiation [13]. Due to a consequence of quantum field theory, one experiences a temperature $T = \frac{\hbar a}{2\pi c k_b}$ in an accelerating reference frame; this is known as the Unruh effect [30]. Since a stationary observer asymptotically close to the horizon of a black hole must have such an acceleration, it can be calculated that black holes have a surface temperature given by $T = \frac{1}{8\pi M}$, where M is the mass of the black hole [6].

This thermal entropy allowed for confirmation of the Bekenstein-Hawking entropy formula, but it also did something more serious: from Hawking's assumption of the "No-hair" theorem, this radiation must therefore be invariant of the previous infalling matter which gave mass to the black hole. Therefore, Hawking and Bekenstein's conclusions implied that black holes permanently destroyed the information inside.

3.2. *The information paradox*

The above ideas eventually culminated in what is now one of the most notorious paradoxes in modern physics: the black hole information paradox. According to the mathematics, nature was no longer unitary. Here we find a process which could not be reversed; the infinitesimal bijectivity of states was lost, and no such encoded information could ever be retrieved after evaporation.

This means one of two things: either black holes are time-symmetry breaking, or the Hawking radiation emitted must be one to one with the states of infalling matter. Put simply, either the information is or is not lost forever. Attempts to resolve this paradox have led to various proposals which challenge the foundational assumptions of quantum theory and general relativity. One of the most prominent ideas is the black hole complementarity principle, proposed by Leonard Susskind and colleagues [27]. This principle suggests that information is both reflected at the event horizon and passed into the black hole interior, with no contradiction for an outside observer. However, this proposal has faced criticism and challenges in such forms as the "firewall" paradox, which argues that an infalling observer would experience a wall of high-energy particles at the event horizon, destroying the information on the boundary[1].

In the final section, we will return to this paradox with new insights, providing a brief overview of the current understanding of unitarity and black holes.

[4] Thirty years later, Juan Maldacena would show a correspondence between Supersymmetric Yang-Mills theories in AdS_n space and CFTs defined on the n-1 dimensional boundary of such a space [20]. This suggests then no information paradox, as the black hole boundary can evolve while unitarily, preserving the information inside holographically. This correspondence roots much of the further analysis on black hole information, and is fundamental in the mainstream view on the modern interpretation of black hole information.

3.3. *Limitations of QFT: Quantum Gravity*

Many of the problems which arose in pushing QFT to its limits also came up in attempts to resolve the black hole information paradox, ultimately leading physicists to approach the issue with a broader picture in mind. Clearly, if there was going to be any semblance of quantum gravity, QFT was not enough. This is where string theory began to enter the mainstream conversation, eventually becoming one of the most comprehensive and analytically-sound "theories of everything", quantum gravity and all.

3.4. *The Kaluza-Klein mechanism*

To understand the nature of string theory, it is illuminating to begin where it did, with Kaluza-Klein compactification. In 1921, Theodor Kaluza postulated that there was, in addition to the standard Minkowski metric, a 5th dimension of spacetime [18]. Consider for a moment the standard spacetime coordinates $\{x^0, x^1, x^2, x^3\}$, along with a fifth coordinate, x^4, which has the identification:

$$x^4 \simeq x^4 + 2\pi R \tag{3.3}$$

This has the physical interpretation of "compactifying" the dimension: Going around a distance $2\pi R$ returns one to the starting point. Considering a line element in this space, we write the metric as:

$$ds^2 = G_{\mu\nu} dx^\mu dx^\nu + G_{44}(dx^4 + A_\mu dx^\mu)^2 \tag{3.4}$$

Where $\{\mu, \nu\}$ run over the typical Minkowski metric. Note that we identify the vector A_μ to the off-diagonal terms which come from the addition of the compactified dimension. This will become the typical, semiclassical electromagnetic potential. Note that under this framework, reparameterizations of the $(3+1)$-Minkowski space of the form $x'^\mu(x^\nu)$, x^4 transforms as

$$x^{'4} = x^4 + \lambda(x^\nu) \tag{3.5}$$

Subsequently, our four-vector A_μ transforms as

$$A'_\mu = A_\mu - \partial_\mu \lambda \tag{3.6}$$

Inheriting $U(1)$ gauge symmetry. So gauge transformations seem to arise as part of the higher-dimensional coordinate group. This is the *Kaluza-Klein* mechanism. From here, we can expand an arbitrary massless scalar field ϕ in terms of eigenmodes in x^4, since single-valuedness in this dimension gives boundary conditions which lead to quantization. Expanding, we have:

$$\phi(\mathbf{x}) = \sum_{n=-\infty}^{\infty} \phi_n(x^\mu) e^{inx^d/R} \tag{3.7}$$

In the low-energy regime , only the zero mode is visible; here we see something incredible: the Kaluza-Klein geometry gives rise to general relativity equipped with an electrodynamic field, $U(1)$ gauge symmetry and all. Consider now a conformal field theory compactified on some background worldsheet X[5], such that $X \cong X + 2\pi R$. As we have just seen, the requirement for the operator which translates strings once around the circle to be single-valued leads to a quantization of the momentum,

$$k = \frac{n}{R} \tag{3.8}$$

$$n \in \mathbb{Z} \tag{3.9}$$

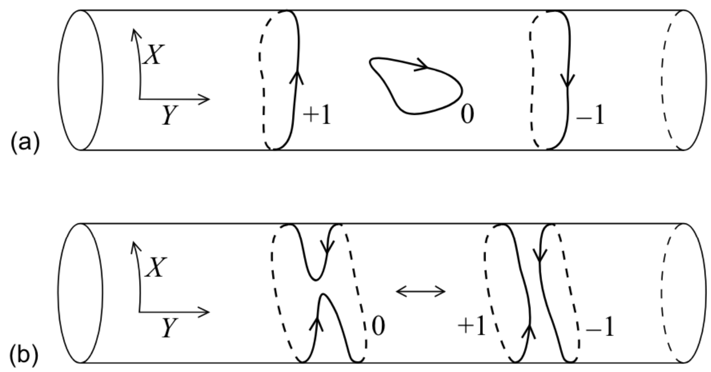
[5] A worldsheet is the background on which strings propagate. They are constrained by Conformal Field Theories, which we will cover soon

However, invariance under conformal transformations does not necessarily require that a function defined on a compact dimension be single-valued: This is an effect not found in quantum field theories. Because the string can "wind around" the compact dimension of space, we can have the following condition:

$$X(\sigma + 2\pi) = X(\sigma) + 2\pi R w \tag{3.10}$$

$$w \in \mathbb{Z} \tag{3.11}$$

This integer w is known as the *winding number*, and it corresponds to a topologically nontrivial orientation of a string on a compact dimension. Just as $\pi_1(S_1) \cong \mathbb{Z}$, string orientations can be classified by a double sum over both the momentum and winding number, both of which are quantized.



The propagation of closed strings along a compact dimension. The winding number w is conserved in string interactions, and generates the richness of the mass spectrum of string states. Here, time is associated with the Y-direction.

4. BASICS OF STRING THEORY

By the 1960s, Kaluza-Klein theory had been developed further, and more fundamental results from the early precursors to string theory had began to crystallize into a theory much deeper than QFT; one with none of the gauge fixing or experimental renormalizations. This was string theory.

At its core, string theory is defined on a "worldsheet", $X(\sigma, \tau)$, where σ is taken to be spatial and τ temporal. Since string theory is formulated on the idea that all matter is made up of vibrating strings which are either "open" or "closed", the theory requires the action on the worldsheet to possess, in addition to the usual Poincaré invariance from General Relativity, invariances under diffeomorphisms and local rescalings (Weyl invariance). This is known as a Conformal Field Theory (CFT), and it forms the background for the dynamics of strings[6].

4.1. *The Nambu-Goto and Polyakov actions*

To start, we begin with the most basic Poincaré-invariant action,

$$S_{NG} = -T \int d\tau d\sigma \sqrt{-\det(\partial_a X^\mu \partial_b X_\mu)} \tag{4.1}$$

Which is known as the Nambu-Goto action [22]. Introducing a background metric γ_{ab} on the worldsheet allows us to write a more useful, equivalent form of this action functional known as the Polyakov action. Again, by the D-dimensional Poincaré symmetry group and the diffeomorphism invariance, we are able to add a background "auxillary" metric:

$$S = -\frac{T}{2} \int d\tau d\sigma \sqrt{-h} \left(h^{ab} \partial_a X^\mu \partial_b X_\mu + \lambda(h_{ab} h^{ab} - 2)\right) \tag{4.2}$$

[6] Conformal invariance is in fact a very deep topic, with applications from statistical mechanics to QFT. In string theory, it arises out of a need for the Polyakov action to preserve scale invariance at a quantum level. This is also where the "magic" number of $D = 26$ dimensions comes from.

Here, $h = \det(h_{ab})$, and λ is a Lagrange multiplier that enforces the constraint $h_{ab}h^{ab} = 2$. Integrating out λ leads back to the Nambu-Goto action. To make the Weyl invariance manifest, we rescale the metric $h_{ab} \to e^{2\phi}h_{ab}$ and choose the conformal gauge $h_{ab} = e^{2\phi}\eta_{ab}$, where η_{ab} is the flat metric. The action becomes:

$$S = -\frac{T}{2}\int d\tau d\sigma \left(\partial_a X^\mu \partial^a X_\mu + 2\lambda\right) \tag{4.3}$$

The equation of motion for λ implies that it is a constant, which can be absorbed into a shift of the action by a constant. Defining the parameter[7] $\alpha' = (2\pi T)^{-1}$, we obtain the Polyakov action:

$$S_P = -\frac{1}{4\pi\alpha'}\int d\tau d\sigma \sqrt{-h} h^{ab}\partial_a X^\mu \partial_b X_\mu \tag{4.4}$$

The Polyakov action is classically equivalent to the Nambu-Goto action but has the advantage of being quadratic in the fields X^μ, making quantization more straightforward. The action is invariant under worldsheet diffeomorphisms and Weyl transformations, which are the fundamental symmetries of string theory.

4.2. *Weyl invariance and the critical dimension*

In this picture, the metric can be understood as the local change in action under variation in the background metric. This comes from varying the action with respect to the worldsheet metric h_{ab} to obtain the energy-momentum tensor T_{ab}:

$$\delta S_P = -\frac{1}{4\pi\alpha'}\int d\tau d\sigma \sqrt{-h}\delta h^{ab}\left(\partial_a X^\mu \partial_b X_\mu - \frac{1}{2}h_{ab}h^{cd}\partial_c X^\mu \partial_d X_\mu\right) \tag{4.5}$$

Defining the energy-momentum tensor as:

$$T_{ab} = -\frac{2\pi}{\sqrt{-h}}\frac{\delta S_P}{\delta h^{ab}} \tag{4.6}$$

We find:

$$T_{ab} = -\frac{1}{\alpha'}\left(\partial_a X^\mu \partial_b X_\mu - \frac{1}{2}h_{ab}h^{cd}\partial_c X^\mu \partial_d X_\mu\right) \tag{4.7}$$

The equation of motion for the worldsheet metric, obtained by setting $\delta S_P/\delta h^{ab} = 0$, implies that the energy-momentum tensor vanishes ($T_{ab} = 0$).

This condition is known as the Virasoro constraint and is a consequence of the diffeomorphism and Weyl invariance of the Polyakov action.

Now, let's consider the variation of the Polyakov action with respect to the fields X^μ:

$$\delta S_P = -\frac{1}{2\pi\alpha'}\int d\tau d\sigma \sqrt{-h}\nabla^a\left(\partial_a X^\mu\right)\delta X_\mu + \frac{1}{2\pi\alpha'}\int d\tau d\sigma \sqrt{-h} n^a \partial_a X^\mu \delta X_\mu \tag{4.8}$$

Here, n^a is the unit normal vector to the boundary of the worldsheet. The first term in the variation gives the equation of motion for X^μ, while the second term is a boundary term that must vanish for the variational principle to be well-defined.

4.3. *A purely-topological action*

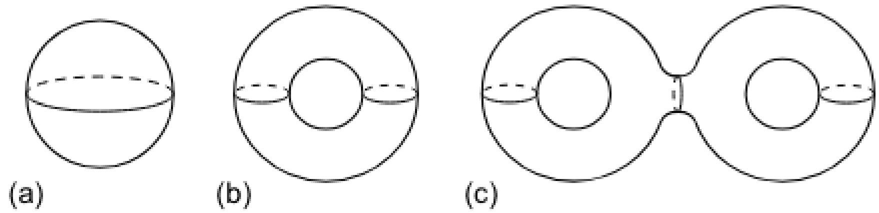
To ensure that the boundary term vanishes, we can impose either Neumann or Dirichlet boundary conditions on the fields X^μ.[8] Neumann boundary conditions allow the string endpoints to move freely, while Dirichlet boundary conditions fix the string endpoints to specific points in spacetime. The coordinate invariance and Poincaré invariance then allow for one additional term beyond the original S_P:

$$\chi = \frac{1}{4\pi}\int d\tau d\sigma \sqrt{-h} R \tag{4.9}$$

[7] Here, T is the tension in the string. Though it can be interpreted in the same way as the tension in a classical string, it is more akin to the relative weighting of the action over the worldsheet.

[8] Neumann: $n^a \partial_a X^\mu|_{\partial\sigma} = 0$ Dirichlet: $\delta X^\mu|_{\partial\sigma} = 0$, where n^a is normal to the boundary ∂M.

Here, R is the Ricci scalar curvature of the worldsheet. The topological term does not depend on the fields X^μ and, therefore, does not contribute to their equations of motion. However, it plays the same role as does the operator-product expansion in quantum field theory. In string theory, we sum over topologies rather than operator expansions.



The first three compact orientable manifolds of genus 0, 1, and 2, respectively.[9]

Since χ is allowed by the symmetries, we attach it to S_p with a free parameter λ to control the magnitude. This will, as it turns out, determine the string coupling constant g_s.

This gives us a generalized action S :

$$S = S_X + \lambda\chi \tag{4.10}$$

$$S_X = \frac{1}{4\pi\alpha'} \int_M d\tau d\sigma \sqrt{g} g^{ab} \partial_a X^\mu \partial_b X_\mu \tag{4.11}$$

$$\chi = \frac{1}{4\pi} \int_M d\tau d\sigma \sqrt{g} R + \frac{1}{2\pi} \int_{\partial M} ds k \tag{4.12}$$

This action is diffeomorphically invariant, Weyl invariant, and Poincaré invariant with an expansion over topologies. We also include the integral over the boundary of the manifold M for the case of a noncompact sum. For the purposes of this introduction to String theory, however, we limit our scope to worldsheets over compact manifolds, in which case this term is zero.

From the Gauss-Bonnet theorem [12], the χ term in the action can be understood as a purely topological artifact: it depends only on the Euler number of the surface $\chi(\mathcal{M})$:

$$\int_{\mathcal{M}} d\tau d\sigma \sqrt{g} R = 2\pi\chi(\mathcal{M}) \tag{4.13}$$

Where $\chi = 2 - 2g - b - c$; corresponding to the number of handles, boundaries, and cross-caps, respectively. For closed compact orientable[10] manifolds, we consider only varying g, however, in other formulations of string theory, summations over a broader class of topologies include manifolds with nonzero values for b and c, such as a Klein bottle or a Möbius strip.

In terms of the path integral formulation (where we consider a functional $e^{iS_{cl}/\hbar}$), each successive topological structure[11] contained in the sum is mediated by λ:

$$g_0^2 \sim g_c \sim e^\lambda \tag{4.14}$$

4.4. *Vertex Operators and Weyl Invariance*

Just as in any standard quantum theory, String theory makes use of vertex operators as the main functorial component from which to calculate tangible probabilities. The purpose of this section is to derive intuition behind these, as well as to demonstrate the power of quantum Weyl invariance in generating a consistent quantum theory of gravity. However, because of the diffeomorphism—Weyl invariance of the theory, there are only a small number of possible configurations of the dynamical fields. In maximum generality, these must mathematically satisfy

[9] Photo from Polchinski's *Introduction to the Bosonic String*

[10]

[11] Loosely speaking, the relationship between g_0 and g_c is quadratic is because of the dependence of χ on each successive topological structure, and this set depends on if you count non-compact topologies.

$$V_1 = g_c \int d^2\sigma \sqrt{g} \left(\frac{\alpha'}{2} (g^{ab} s_{\mu\nu} + i\epsilon^{ab} a_{\mu\nu}) \left[\partial_a X^\mu \partial_b X^\nu e^{ik\cdot X} \right]_r + \alpha' \phi R \left[e^{ik\cdot X} \right]_r \right),$$

Where $s_{\mu\nu}, a_{\mu\nu}$, and ϕ are a symmetric matrix, antisymmetric matrix, and a constant, respectively. These correspond to the graviton, electromagnetic field, and dilaton, though we have yet to show this.

Consider the Polyakov action, refitted with a dynamical background metric $G_{\mu\nu}$:

$$S_\sigma = \frac{1}{4\pi\alpha'} \int_M d^2\sigma \sqrt{g} g^{ab} G_{\mu\nu}(X) \partial_a X^\mu \partial_b X^\nu. \tag{4.15}$$

Remember, since we are working in a quantum theory of gravity, gravity is defined in terms of a vertex operator just like any other particle. Since $G_{\mu\nu}$ is then not a background but a dynamical field, we can't just substitute in the metric. Consider instead a small perturbation from Minkowski space, where

$$G_{\mu\nu}(X) = \eta_{\mu\nu} + \chi_{\mu\nu}(X) \tag{4.16}$$

Then, in the path integral formulation,

$$\exp(-S_\sigma) = \exp(-S_P) \left[1 - \frac{1}{4\pi\alpha'} \int_M d^2\sigma \sqrt{g} g^{ab} \chi_{\mu\nu}(X) \partial_a X^\mu \partial_b X^\nu + \dots \right]. \tag{4.17}$$

Substituting χ for s, this corresponds to the graviton field above; the relation is through the vertex operator

$$\chi_{\mu\nu}(X) = -4\pi g_c e^{ik\cdot X} s_{\mu\nu}. \tag{4.18}$$

Without too much mess, we can generalize this comparison between a small perturbation in the background metric and the associated vertex operators. These are generally the same form, with the only difference resulting from the different matrices associated with each field:[12]

$$G_{\mu\nu}(X) = \eta_{\mu\nu} - 4\pi g_c s_{\mu\nu} e^{ik\cdot X} \tag{4.19}$$

$$B_{\mu\nu}(X) = -4\pi g_c a_{\mu\nu} e^{ik\cdot X} \tag{4.20}$$

$$\Phi(X) = -4\pi g_c \phi e^{ik\cdot X}. \tag{4.21}$$

If we re-fit the original equation with the analogous vertex-operator forms of the electromagnetic and dilaton fields, we can write the action over a dynamical background as

$$S_\sigma = \frac{1}{4\pi\alpha'} \int_M d^2\sigma \sqrt{g} \Big[g^{ab} G_{\mu\nu}(X) + i\epsilon^{ab} B_{\mu\nu}(X) \Big] \partial_a X^\mu \partial_b X^\nu + \frac{1}{4\pi} \int_M d^2\sigma \sqrt{g} \alpha' R \Phi(X).$$

4.5. *Equations of Motion*

To derive equations of motion from this action functional, remember that Weyl invariance still holds. That is, our metric must be traceless[13]. Taking the metric to be of the above form, it is a straightforward–though exceptionally arduous–task to show that the trace of T is proportional to the following set of beta functions:[14]

[12] This derivation is given in Polchinski, Chapter 3.7

[13] This comes from Weyl invariance, since rescaling scales the trace, so invariance implies tracelessness.

[14] These beta functions are in fact very similar to those in QFT

$$T_a^a = -\frac{1}{2\alpha'}\beta^G_{\mu\nu}g^{ab}\partial_a X^\mu \partial_b X^\nu - \frac{i}{2\alpha'}\beta^B_{\mu\nu}\epsilon^{ab}\partial_a X^\mu \partial_b X^\nu - \frac{1}{2}\beta^\Phi R \tag{4.22}$$

$$\beta^G_{\mu\nu} \approx -\frac{\alpha'}{2}\partial^2\chi_{\mu\nu} - \partial_\nu\partial^\omega\chi_{\mu\omega} - \partial_\mu\partial^\omega\chi_{\omega\nu} + \partial_\mu\partial_\nu\chi^\omega_\omega + 2\alpha'\partial_\mu\partial_\nu\Phi \tag{4.23}$$

$$\beta^B_{\mu\nu} \approx -\frac{\alpha'}{2}\partial^\omega(\partial_\omega B_{\mu\nu} + \partial_\mu B_{\nu\omega} + \partial_\nu B_{\omega\mu}) \tag{4.24}$$

$$\beta^\Phi \approx \frac{D-26}{6} - \frac{\alpha'}{2}\partial^2\Phi. \tag{4.25}$$

Therefore, from only Weyl invariance of the trace of the energy-momentum tensor, we require that

$$\beta^G_{\mu\nu} = \beta^B_{\mu\nu} = \beta^\Phi = 0 \tag{4.26}$$

The scope of this equation is colossal; perhaps written in a more familiar form they will be more recognizable:

$$G_{\mu\nu} = 8\pi T_{\mu\nu} \tag{4.27}$$

$$\partial^\omega(\partial_\omega B_{\mu\nu} + \partial_\mu B_{\nu\omega} + \partial_\nu B_{\omega\mu}) = 0 \tag{4.28}$$

$$(\partial_\mu\partial^\mu - m)\Phi = 0 \tag{4.29}$$

The consequence of Weyl invariance in an operator-formalized string theory is Einstein's field equation, Maxwell's equations in a vacuum, and the Klein-Gordon equation. Now, let us turn our attention to attempt to calculate something with all of this generality.

5. CALCULATING SCATTERING AMPLITUDES

We now turn our attention to grasping the fundamentals of string dynamics and how their theoretical framework is actually implemented to calculate something as tangible as a scattering amplitude. The approach is similar to that of QFT, though instead of an algebraic sum ordered by operator combinations, the string theory S-matrix sums over topologies; orientability and compactness requirements depend on the type of theory considered.

5.1. *The path integral formulation*

Since we want our scattering amplitude to be independent of diffeomorphisms and Weyl transformations, we ought to choose a "representative element" from the total space of a given topological class and divide by the local density of total possible configurations under the full collection of invariant transformations. This is given by the correlation function in the 2-dimensional CFT on the worldsheet:

$$S_{j_1,...,j_n}(k_1,...,k_n) = \sum_{\mathcal{C}} \frac{[dX dg]}{V_{Diff\times Weyl}} e^{-S_X - \lambda\chi} \prod_{i=1}^{n} d^2\sigma_i \sqrt{g(\sigma_i)}\mathcal{V}_{j_i}(k_i,\sigma_i) \tag{5.1}$$

Where $S_{j_1,...,j_n}(k_1,...,k_n)$ represents the scattering amplitude, $\mathcal{C}$ represents the space of topologies (in the case of Bosonic string theory, these are categorized solely by the number of "handles"), and the product over $\mathcal{V}_{j_i}(k_i,\sigma_i)$ includes the vertex operators eludicated to in the previous section on CFT. We integrate over dX, the fields, and dg, the metric space. Though this integral is actually an integral over the gauge group, the moduli space, and the unfixed positions of the operators,[15] we will consider the "tree-level" amplitudes over the sphere, which has a trivial moduli space.

For the case of the sphere, we can simplify the above formula down to a scattering amplitude which only depends on a few parameters:

$$\mathcal{A}^{(n)} = \frac{1}{g_s^2}\frac{1}{Vol}\int [dX dg] e^{-S_X} \prod_{i=1}^{n} \mathcal{V}_{j_i}(p_i) \tag{5.2}$$

[15] i.e. $[dgdX] \simeq [dg]d^{2n}\sigma \to [d\zeta]d^\mu t d^{2n-k}\sigma$, where μ runs over the moduli space, ζ the Weyl-diffeomorphism space, and σ the topological space

5.2. *The four-point amplitude*

To deal with this daunting integral, we begin by gauge-fixing the diffeomorphism-Weyl invariance. That is, we "fix" the integral over a particular initialization of the space of possible metrics. For example, we can use the stereographic projection onto the Riemann sphere:

$$ds^2 = \frac{4R^2}{(1+|z|^2)^2} dz d\bar{z} \tag{5.3}$$

This looks nice, but it does not fix the gauge symmetry entirely. We still have free parameters which result from combining Weyl transformations and diffeomorphisms. This is known as the conformal group over the Riemann sphere. This is the group of fractional linear transformations of the form

$$z \to \frac{\alpha z + \beta}{\gamma z + \delta} \tag{5.4}$$

Where $\alpha\delta - \beta\gamma = 1$. This has a symmetry group isomorphic to $PSL(2;\mathbb{C})$. Reducing the above amplitude expression to the following:

$$\mathcal{A}^{(n)} = \frac{g_s^{m-2}}{PSL(2;\mathbb{C})} \int [dXdg] e^{-S_X} \prod_{i=1}^{n} d^2 z_i \langle \hat{V}(z_1,p_1), ..., \hat{V}(z_m,p_m) \rangle \tag{5.5}$$

$$\mathcal{A}^{(n)} \sim \frac{g_s^{m-2}}{PSL(2;\mathbb{C})} \int [dXdg] e^{-S_X} \delta^{26}(\sum_{i=1}^{m} p_i) \prod_{i=1}^{n} d^2 z_i e^{\frac{\alpha'}{2}\sum_{j,l} p_j \cdot p_l \cdot ln|z_j - z_l|} \tag{5.6}$$

$$\mathcal{A}^{(n)} \sim \frac{g_s^{m-2}}{PSL(2;\mathbb{C})} \int [dXdg] e^{-S_X} \delta^{26}(\sum_{i=1}^{m} p_i) \prod_{i=1}^{n} d^2 z_i \prod_{j<l} |z_j - z_l|^{\alpha' p_j \cdot p_l} \tag{5.7}$$

Where we have substituted the general expression for the vertex operator product for the correlation function. In the two steps above, all we have done is used the aforementioned CFT techniques to simplify the operator product into a correlation function. All that remains is to fix three of the vertex operators on the Riemann sphere to fix the residual $PSL(2,\mathbb{C})$ symmetry we are dividing by[16]. We have also added a $\delta^{26}(\sum_{i=1}^{m} p_i)$ to account for the conservation of momentum.[17]

Lets compute the "four-point" amplitude for such a system. Since we have the freedom to choose three of the four points on the Riemann sphere which to act the vertex operators, we abuse the conformal freedom and fix z_1, z_2, z_4:

$$z_1 = \infty, z_2 = 0, z_3 = z, z_4 = 1 \tag{5.8}$$

Using the above formula, this gives us the following integral:

$$\mathcal{A}^{(4)} \sim g_s^2 \delta^{26}(\sum_i p_i) \int d^2 z |z|^{\alpha' p_2 \cdot p_3} |1-z|^{\alpha' p_3 \cdot p_4} \tag{5.9}$$

As it is calculated in [28], this is equal to:

$$\mathcal{A}^{(4)} \sim g_s^2 \delta^{26}(\sum_i p_i) \frac{\Gamma(-1 + \frac{(p_1+p_2)\alpha'}{4})\Gamma(-1 + \frac{(p_1+p_3)\alpha'}{4})\Gamma(-1 + \frac{(p_1+p_4)\alpha'}{4})}{\Gamma(2 - \frac{(p_1+p_2)\alpha'}{4})\Gamma(2 - \frac{(p_1+p_3)\alpha'}{4})\Gamma(2 - \frac{(p_1+p_4)\alpha'}{4})} \tag{5.10}$$

Though it is messy, this formula gives us something tangible about how strings interact. Given four strings, we can calculate their exact scattering given initial momenta. Interestingly, this amplitude contains many—in fact infinitely many—poles. These poles occur at values corresponding to successively more massive modes of the string, corresponding to the infinite set of tree-level, QFT-style field-theory diagrams.

[16] In this context, we really mean that we divide by the Fadeev-Popov invariant, which is a local measure of the density of possible configurations, but the idea is the same.

[17] This comes from the CFT algebra, particularly the singularity of the $\frac{1}{\partial\bar{\partial}}$ operator at the origin. It comes from the integral over the "zero mode" in the path integral formulation; $\int e^{\sum_{i=1}^{m} p_i \cdot x} dx \sim \delta^{26}(\sum_{i=1}^{m} p_i)$

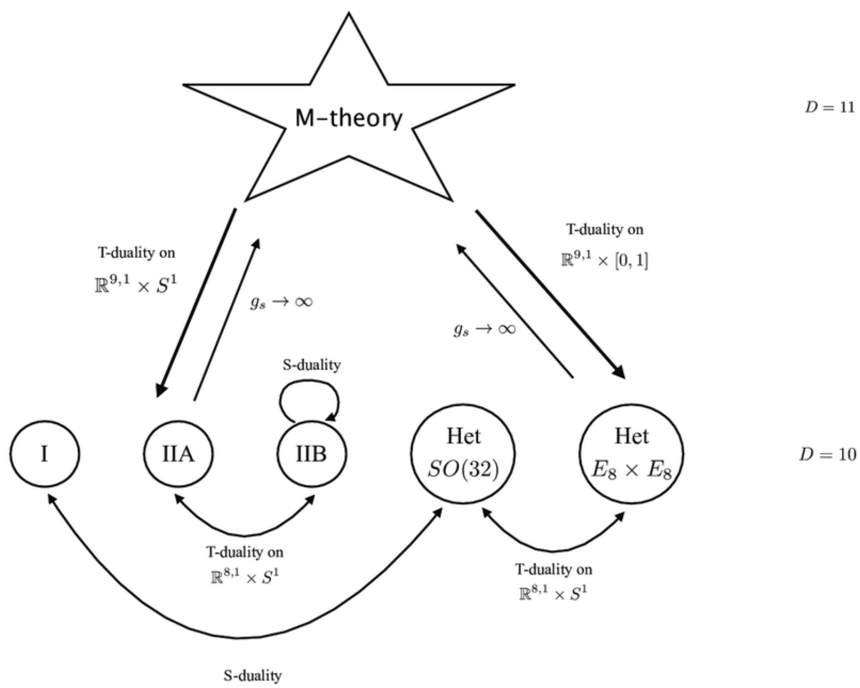
5.3. *A Network of Dualities*

As discussed above in our derivation of the Klazua-Klein mechanism, one of the distinguishing features of string theory is its requirement of multiple internal compactified dimensions. For the sake of simplicity, we have—and will continue to examine—the new effects which mostly result from only one added dimension, but this is only a toy model of real string theory. This will be a good time to address the elephant in the room: All of our derivations and observations are framed in the context of Bosonic string theory, which is largely a toy model useful for conceptualization.

5.4. *M-theory*

The modern string theory universe is much more general. In reality, Bosonic, or "Type I" string theory, is just one limiting case of a more general theoretical framework known as "M-theory" [31]. This is a unique framework in which all formulations of string theory exist as limiting cases. Below is a diagram representing the relationship of the various formulations of string theory to the generalized M-theory, as well as a chart depicting the relationships between the various limiting cases in terms of various dualities. These dualities, known as "S" and "T" dualities, transform limiting cases of M-theory between one another.

S-duality relates strong and weak coupling regimes in string theory through the relation $g_s \leftrightarrow 1/g_s$, where g_s is the string coupling constant [11]. If a string theory with coupling g_s is strongly coupled, making the perturbation theory ill-defined, its S-dual description with coupling $1/g_s$ is weakly coupled, allowing perturbative calculations. Mathematically, an S-duality transformation acts on the dilaton field[18] ϕ as $\phi \to -\phi$, inverting the string coupling $g_s = e^{\langle \phi \rangle}$. T-duality relates string theories compactified on circles of radius R and α'/R, where α' is a scale parameter. Studying strings on a circle of radius R is equivalent to studying a different string theory on a circle with radius α'/R [9]. This can be seen by performing a Fourier decomposition of string coordinates, interchanging winding and momentum modes under $R \to \alpha'/R$. These dualities relate the five superstring theories in 10 dimensions and 11-dimensional M-theory, suggesting they are different limits of a single underlying theory.



A depiction of the various limiting cases of M-theory, where each limiting case corresponds to a string theory, and each string theory is related within a network of dualities.

There are many questions that can be asked about the implications and meaning of this structure in the context of the black hole information paradox. In the previous sections, we reconstructed Bosonic string theory—notice that it is

[18] The dilaton field ϕ is a scalar field that appears in theories of gravity, such as string theory, and represents the local size of extra spatial dimensions. It couples to matter fields, affecting the strength of gravity, and leading to interesting phenomena such as the variation of fundamental constants and the modification of gravitational interactions.

not present in this diagram. That is because this string theory is not complete; it does not include Fermionic matter interactions and is incomplete; the various methods of supersymmetric reduction or Lie algebra embeddings are what generate the various string theories, and they are not necessary unique from one another as implied by the dualities.

6. THE STRING-BLACK HOLE CORRESPONDENCE

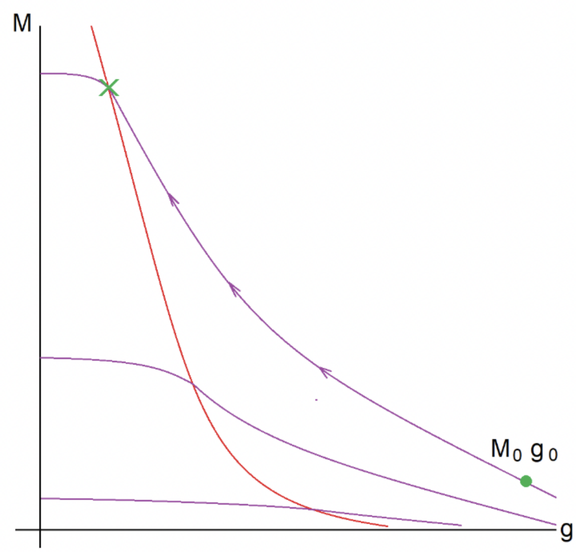
As a locally well-defined "theory of everything", it is necessary that string theory reduce to macroscopic semiclassical behavior at limiting scales. Since semiclassical black hole thermodynamics were well understood by the 1990s, this was the standard to which stringy formulations of black holes would have to reduce to in order to be consistent with Hawking and Bekenstein's semiclassical image of black holes.

On one end of the spectrum were self-gravitating string states, which could be understood via string Hamiltonians and the worldsheet dynamics outlined above, and one the other end were black holes sufficiently cool below any critical string behavior, for which stringy corrections to any observables were far too small to observe. How were these two solutions supposed to be connected?

6.1. *Susskind's stretched horizon*

In 1993 at a conference at Rutgers, Leonard Susskind supposed that there was a way to connect the two, and it came from the following spark of inspiration: "Adiabatically vary the string coupling constant, or if you like, the background dilaton field until gravity gets so weak that it can no longer hold the string onto the horizon. When that happens the black hole should disappear and become a collection of almost free strings. If done slowly enough the entropy should not change during the course of the process (technically one would say that entropy is an adiabatic invariant), and we can calculate the original black hole entropy by using free string theory." [26].

This led to back-of-the-napkin calculations which, based on a few assumptions, led to a recovery of the Bekenstein-Hawking entropy relationship from basic principles. Susskind assumed the following: First, assume a black hole of mass M_0 and string coupling g_0 (hence $G_0 = l_s^2 g_0^2$), and vary Mg along the adiabat (constant-entropy) until R_s, or the Schwarzschild radius, is of order unity to l_s.



A representation of the adiabats (shown in purple) for different starting couplets of M_0, g_0. Varying along these curves until they intersect the transition point (red) allows for a calculation of string entropy on black holes.

Hence, from the relationships that

$$M = \frac{g^2}{l_s} \tag{6.1}$$

$$Mg = M_0 g_0 \tag{6.2}$$

We can see that there is a "matching point" (indicated by the green "x" above). This occurs at the point $Ml_s = M_0^2 g_0^2 l_s$, which is the Horowitz-Polchinski correspondence point. This is where the solution is not well understood, and where the black hole-string transition is believed to take place. These results from Susskind directly imply a relationship $S = M_0^2 G_0$, which agrees with the Bekenstein-Hawking entropy formula.

A more rigorous analysis was needed. How are we to understand string entropy? What happens to the strings as they approach the horizon? Consider the picture from a fiducial observer of a Schwarzschild black hole. Infalling strings will "melt onto" the horizon, as they will never be seen to cross. One consequence of this is the incredible density of information—or entropy—which must be contained on the boundary. Because the level density of string states scales exponentially with the mass of a string, one incredible consequence of this will be the likely formation of one incredibly long string. Simply given the relative number of possible microstates, it is a statistical guarantee that such a string configuration forms.

Consider the "stretched horizon" of such a black hole. This is the horizon a distance l_s above the true event horizon, and it is also where the Unruh temperature will, due to the thermal bath of strings, approach the Hagedorn temperature. This temperature is an essential component of string theory, as it is the temperature at which the partition function of a string configuration diverges, indicating a phase transition.

At this temperature, string effects reach order unity to the standard black hole effects, and, due to the redshift at this distance, Susskind was able to define the so-called "Stretched Horizon" energy and temperature,

$$E_{SH} = \frac{2M^2G}{l_s} \tag{6.3}$$

$$T_{SH} = \frac{1}{2\pi l_s} \tag{6.4}$$

Which together lead to the expression for the number of microstates of a black hole of mass M,

$$log(N(E)) \sim El_s \rightarrow log(N(M)) \sim M^2G \tag{6.5}$$

Recovering again the Hawking-Bekenstein entropy form from the string-microstate picture [16].

6.2. *The Horowitz-Polchinski correspondence*

Susskind's pioneering work into a duality between black hole states and string states was well received. In fact, it was the inspiration for Horowitz and Polchinski to draw a more explicit correspondence between black holes and strings [14]. In 1997, they published a paper proved the following:

When the curvature at the horizon of a black hole becomes greater than the string scale, the typical black hole state becomes a typical state of strings and D-branes with the same charges and angular momentum. Furthermore, the mass changes by at most a factor of order unity during the transition.

The goal of Horowitz and Polchinski was relatively simple: To construct a direct comparison between the string and black hole solutions from a generic metric, and to show that a smooth transition between them was possible. We present here the essence of this argument. First, we consider the entropy of a single string. For a bosonic string in D spacetime dimensions, the total level number N is related to the mass M of the string state by:

$$M^2 = \frac{N}{\alpha'}, \tag{6.6}$$

The number of string states at level N, denoted by d(N), grows exponentially with N:

$$d(N) \sim \exp\left(4\pi\sqrt{\frac{N}{6}}\right). \tag{6.7}$$

The entropy of the string state is given by the logarithm of the number of microstates, and thus the entropy of a highly-excited string scales by the square root of the energy level:

$$S_{\text{string}} = \log d(N) \sim 4\pi\sqrt{\frac{N}{6}} \to S_{\text{string}} \sim \sqrt{N}. \tag{6.8}$$

Now, consider a general d-dimensional Schwarzschild metric:

$$ds^2 = -\left(1 - \frac{r_0}{r^{D-3}}\right)dt^2 + \left(1 - \frac{r_0}{r^{D-3}}\right)^{-1}dr^2 + r^2 d\Omega^2_{D-2}, \tag{6.9}$$

Where

$$M_{BH} \sim \frac{r_0^{d-2}}{G}, \quad M_s \sim \frac{N}{\alpha'}, \quad G_N = g^2\alpha' \tag{6.10}$$

as before. Consider the point where $r_0^2 \sim \alpha'$ to be the order of magnitude at which the transition occurs. Setting the masses equal then leaves

$$M_{BH}^2 \sim \frac{r_0^{d-2}}{G^2} \sim \frac{N}{\alpha'} \tag{6.11}$$

From which we can use the fact that, in D dimensions, the Hawking-Bekenstein entropy formula gives the relation $S_{BH} \sim \frac{r_0^{d-1}}{G}$. Then we conclude:

$$S_{BH} \sim \frac{r_0^{d-1}}{G} \sim \frac{\alpha^{\frac{d-1}{2}}}{G} \sim \sqrt{\alpha' \frac{\alpha'^{d-2}}{G^2}} \sim \sqrt{N} \tag{6.12}$$

Hence we derive a black hole entropy which agrees with string entropy to order unity at the correspondence point. This derivation was shown for Schwarzschild black holes, but it also holds for charged (Ramond-Ramond or Neveu-Schwarz) [15] black holes, as well as extremal Reissner-Nordstom black holes, as shown more rigorously by Strominger and Vafa only a year later [25].

7. SELF-GRAVITATING SOLUTIONS

The correspondence principle proposed by Susskind (and formalized by Horowitz and Polchinski) provides ample impliration for a solution which interpolates between the semiclassical and stringlike regimes. Such a solution could potentially provide an analytic connection between strings and black holes, offering a resolution of the information paradox.

Horowitz and Polchinski, in reference to their previous correspondence principle, suggest that there might be a natural way to connect the solution of a string to the solution of a classical black hole [17]. Just as Susskind had found using adiabats, they conjectured that this transformation would occur at a "critical coupling" value, $g_c \sim N^{-1/4}$. However, a string in this regime will scale in size only as allowed by self-interactions. Suprisingly, this "self-interaction" term becomes significant at a coupling $g_0 \sim N^{(d-6)/8}$, and thus for $d = 3$, these interactions must be considered given $g_0 < g_c$. As the coupling is increased from g_0 to g_c, the string size then scales as

$$\ell \sim \frac{\alpha'^{1/2}}{g^2 N^{1/2}}. \tag{7.1}$$

Decreasing from the random-walk scale to the string scale at g_c.

7.1. *Free string states*

The expectation value of an observable X in a typical string state of mass M is given by:

$$\langle X \rangle = Z^{-1}\text{Tr}(Xe^{-\beta H}), \quad Z = \text{Tr}(e^{-\beta H}). \tag{7.2}$$

From above, the number of states of an excited string, $n(M)$, scales as $n(M) \sim e^{\beta_H M}$. Now it is clear why we have a critical temperature β_H: The behavior as $\beta \to \beta_H$ corresponds to the "critical compactification radius" of $R_H = \frac{\beta_H}{2\pi}$. For values of $\beta < \beta_H$, the fundamental winding mode becomes tachyonic:

$$m^2(\beta) = \frac{\beta^2 - \beta_H^2}{4\pi^2\alpha'^2} = \frac{R^2 - R_H^2}{\alpha'}^2 . \tag{7.3}$$

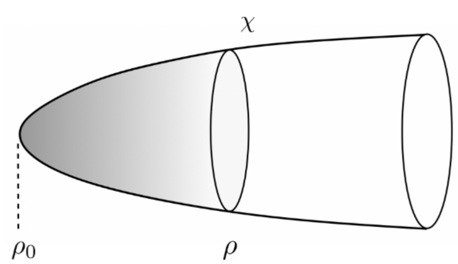
Heuristically, strings that wind multiple times around Euclidean time have a density of states that grows faster than that of singly wound strings. However, they are suppressed by higher powers of the string coupling constant, corresponding to higher genus surfaces in the string perturbation expansion. This suppression is strong enough to make their contribution subdominant compared to the singly-wound strings. This can also be seen in the partition function, where

$$Z = \int [d\chi] e^{-S_\chi} \approx -\sum_a \log \lambda_a \tag{7.4}$$

where λ_a are the eigenvalues of $-\nabla^2 + m(\beta)^2$. In the case of a diverging $\frac{1}{m(\beta)}$ (i.e. small spatial dimensions), the dynamics are governed by the lowest eigenvalue $\lambda_1 = m^2(\beta)$, corresponding to the lowest winding mode. Thus at critical scales the dominating term gives an approximate partition:

$$Z_c(\beta) \approx -\ln \lambda_1 \approx -\ln(\beta - \beta_H). \tag{7.5}$$

The near-horizon geometry can, in the Euclidean-rotated picture, be understood by the "Cigar model", which is used by Chen, Maldacena, and Witten [7] to illustrate the near-horizon geometry on which the string condenste χ is wound:



A Euclidean string worldsheet wrapping the cigar gives rise to the expectation value of the winding mode at radius ρ, where the winding condensate depends on ρ: $\chi(\rho) \propto e^{-(\rho-\rho_0)R}\alpha'$ for $\rho >> \rho_0$, $R >> l_s$.

7.2. *Chen-Maldacena-Witten paper*

It is now time to consider the complete action in the context of a full supersymmetric string theory. In a 2021 review of Horowitz and Polchinksi's solution-generating procedure, Chen, Maldacena, and Witten explicitly solve quasi-analytic solutions in a supersymmetric regime. They treat the dominating lowest winding mode as a "winding condensate", which can take on complex values in spacetime to account for rotational degrees of freedom.

We start with the mass condition mentioned above, as and assume critical radii which correspond to a spectra of string theories. Denote the winding mode field as χ, and the radius of the compactification circle as Re^{φ}, where R is the asymptotic radius which approaches the limit of validity of our solution outside R_H.

$$m^2 = \frac{R^2 - R_H^2}{\alpha'^2}, \quad R_H^{\text{Bosonic}} = 2\ell_s, \quad R_H^{\text{Type II}} = \sqrt{2}\ell_s, \quad \ell_s \equiv \sqrt{\alpha'} \tag{7.6}$$

$$m^2 = \frac{R^2}{\alpha'^2} + \frac{1}{4R^2} - \frac{R_H^2}{\alpha'^2} - \frac{1}{4R_H^2}, \quad R_H^{\text{Heterotic}} = \left(1 + \frac{1}{\sqrt{2}}\right)\ell_s \tag{7.7}$$

We also must set the conditions for the winding mode and momentum of the strings, which come from the critical behavior as outlined above. These are defined to match the lowest-order contribution to the action, and therefore single out the lowest winding mode. Considering the full action, we include the dilaton field ϕ_d, classical d-dimensional action $\mathcal{R}$, as well as the winding terms and mass-coupling terms, we have:

$$I_d = \frac{1}{16\pi G_N}\int d^dx\sqrt{g}e^{-2\phi_d}\left[-\mathcal{R} - 4(\nabla\phi_d)^2 + (\nabla\varphi)^2 + |\nabla\chi|^2 + m^2(\varphi)|\chi|^2\right]. \tag{7.8}$$

For the regime of validity of this action, we expand the mass in terms of powers of φ. This gives the following:

$$m^2(\varphi) = m_\infty^2 + \frac{\kappa}{\alpha'}\varphi + o(\varphi^2), \quad m_\infty^2 \sim \frac{\kappa(R-R_H)}{\alpha' R_H}. \tag{7.9}$$

Where values of κ depend on the quality of string theory:

$$\kappa_{\text{Bosonic}} = 8, \quad \kappa_{\text{Type II}} = 4, \quad \kappa_{\text{Heterotic}} = 4\sqrt{2} \tag{7.10}$$

The first-order term in the expansion of $m^2(\varphi)$ dominates the further terms in the asymptotic regime, so—for the sake of calculation—all other terms are neglected. This action gives us a reference from which we can derive the equations of motion in the usual perturbative manner, resulting in two equations of motion:

$$0 = -\nabla^2\chi + \left(m_\infty^2 + \frac{\kappa}{\alpha'}\varphi\right)\chi, \quad 0 = -2\nabla^2\varphi + \frac{\kappa}{\alpha'}|\chi|^2. \tag{7.11}$$

Rescaling variables to[19]

$$\hat{x} = x\frac{m_\infty}{\sqrt{\zeta}}, \quad \chi(x) = \frac{\alpha'\sqrt{2(d-2)\omega_{d-1}}}{\kappa m_\infty^2\sqrt{\zeta}}\hat{\chi}(\hat{x}) \tag{7.12}$$

Gives a solvable integro-differential equation of the form:

$$-\hat{\nabla}^2\hat{\chi}(x) - \int d^d\hat{y}\frac{|\hat{\chi}(\hat{y})|^2}{|\hat{x}-\hat{y}|^{d-2}}\hat{\chi}(\hat{x}) = -\zeta\hat{\chi}(\hat{x}) \tag{7.13}$$

where ζ is an eigenvalue of the solution for $\hat{\chi}$, which takes its minimum value.[20]

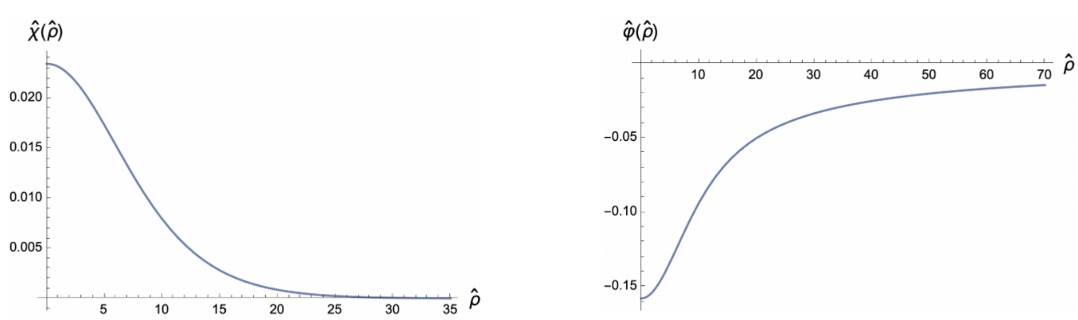
Computing the entropy is relatively straightforward. We take the variation in the action with respect to β, considering the explicit dependence of the Lagrangian with respect to β, since the implicit dependence is accounted for in the equations of motion. This leaves us with a non-zero classical entropy. To first order in $R - R_H$:

$$S = (1-\beta\partial_\beta)log(Z) = \frac{1}{16\pi G_N}\int d^dx\sqrt{g}e^{-2\phi_d}(R\partial_R m^2)|\chi|^2 \tag{7.14}$$

$$\sim \frac{\kappa}{\alpha'\beta_H}\frac{1}{16\pi G_N}\int d^dx|\chi|^2. \sim \frac{2\alpha'(d-2)\omega_{d-1}}{16\pi G_N\kappa\beta_H}\zeta^{\frac{d}{2}-2}m_\infty^{4-d} \sim \frac{(R-R_H)}{2R_H G_N\beta_H\sqrt{\zeta}} \tag{7.15}$$

Where the last equivalence takes $d = 3$. Just like the black hole solution, we are left with a classical entropy, arising from the non-local dependence of the string mass on the radius of the cigar.

7.3. *Explicit solution for d=3*



(a) (b)

[19] and normalizing the total condensate volume to $\int d^d\hat{x}|\hat{\chi}(\hat{x})|^2 = 1$

[20] Hence corresponding to the ground state of the solution (λ_1). More detailed analysis, including negative modes of the solution, can be found in later chapters of [8]

The rescaled solutions for d=3, where (a) represents the winding condensate as a function of radius, and $\hat{\varphi}(\hat{\rho})$ represents (loosely) the Newtonian potential.[21]

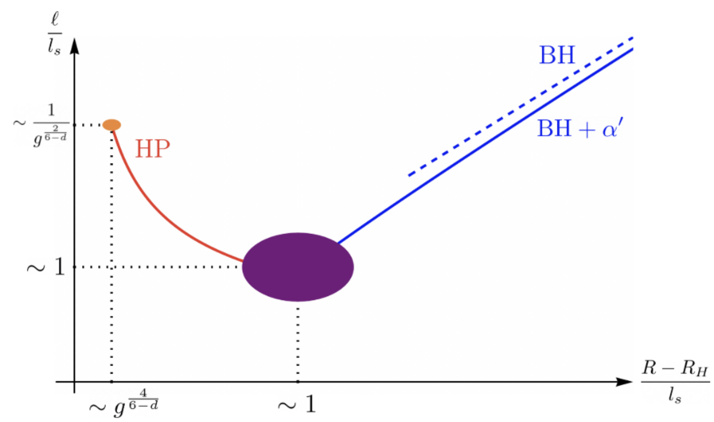
For the case where our solution exists in three noncompact dimensions, we find a value of $\zeta \sim .00813$, with an asymptotic mass given by the above asymptotic expansion. From the relationship of mass to (Hawking) temperature T_H, we can also derive the dependence of the solution's entropy as a function of temperature, and integrate for a full expression:

$$M = \frac{1}{2G_N}\sqrt{\frac{\alpha'(R-R_H)}{\kappa R_H \zeta}} \tag{7.16}$$

$$\frac{dS}{dM} = \beta = \beta_H + \frac{\zeta 8\pi G_N^2}{\kappa R_H \alpha'} M^2 \to S = \beta_H M + \frac{\xi}{3} M^3. \tag{7.17}$$

Therefore, the Horowitz-Polchinski solution gives a positive, cubic correction to the Hawking-Bekenstein equation to leading order.

7.4. *Regime of validity*



Here, the red line is the Horowitz-Polchinksi solution for d=3. It connects the free string picture (the orange oval) and ideally asymptotic to the classical black hole shown in dashed blue[22]. The analytic intractability of the solution is represented by the purple oval.

The solution is valid for small values of $(R - R_H)/\ell_s$. When quantum fluctuations dominate, the solution breaks down, leading to the condition:

$$\frac{R-R_H}{\ell_s} \gtrsim g^{\frac{4}{6-d}}. \tag{7.18}$$

Beyond this regime, understanding smaller values of $R - R_H$ can only come through the free-string picture, which gives

$$S \sim \beta_H M, \quad \ell \sim \ell_s \sqrt{\ell_s M} \tag{7.19}$$

For the free string. Here the entropy is derived from the random-walk picture for sufficiently low-coupling.

8. HOLOGRAPHY, FUZZBALLS, AND BEYOND

In the past two decades, string theorists have made significant progress in understanding black hole microstates and resolving the information paradox.

[21] The true representation is a combination of the dilaton field and the Newtonian potential, related to φ through $e^{-2\phi_d} = e^{-2\phi_D}\beta e^{\varphi}; \phi_D = \frac{1}{2}\varphi$, where D is the total number of dimensions.

[22] The authors also include the first α' correction to the classical black hole, which gives a corrected solution in solid blue. These are the corrections which, at small length scales, diverge relative to the classical description.

8.1. *AdS/CFT correspondence*

One of the most powerful developments in theoretical physics has been the AdS/CFT correspondence, first realized by Juan Maldacena in 1997. The correspondence conjectures a duality between a theory of gravity in (d+1)-dimensional Anti-de Sitter space (AdS_{d+1}) and a conformal field theory (CFT_d) defined on the d-dimensional boundary of the AdS space. For our use, this duality can be expressed in terms of the equivalence of the partition functions:

$$Z_{string}[\partial AdS_{d+1}] = Z_{CFT_d}[\partial AdS_{d+1}] \tag{8.1}$$

where Z_{string} and Z_{CFT} are the partition functions of the string theory and the CFT, respectively. This implies that the physics within the AdS "bulk" space is defined by the information holographically encoded on the boundary. Many speculate that this principle can be applied to the boundary of a black hole, preserving unitarity on the boundary and therefore resolving the information paradox; A black hole in AdS_{d+1} is dual to a thermal state in the CFT_d evolving dynamically and unitarily on the boundary.

8.2. *The fuzzball proposal*

Another string-theoretic approach to the problem is known as the fuzzball proposal, spearheaded by Samir Mathur and collaborators [21]. In this picture, black holes are understood as horizonless, stringy "fuzzballs" that have the same mass, charges, and angular momentum as the corresponding black hole. The fuzzball surface is analogous to the stretched horizon, but it is located slightly outside the would-be event horizon, giving rise to a "soft hair" of strings on the boundary. Mathur argues that fuzzballs provide a complete description of black hole microstates and avoid the need for a singular horizon where information would be lost. Microstates are fuzzball configurations, and the Hawking radiation emerges from quantum fluctuations of the fuzzball itself, carrying away the information of the microstate. While explicit fuzzball solutions have been constructed for certain supersymmetric black holes, the general picture remains speculative.

8.3. *Other Approaches*

Other notable proposals for resolving the information paradox include the the ER=EPR conjecture and matrix models [19][2]. All of these resolutions of the paradox presuppose that unitarity is indeed preserved, which has become the dominant view of the paradox itself. As string theory continues to evolve, new insights into the nature of black holes and spacetime at the Planck scale will become necessary. The resolution of this paradox will hopefully suggest a glimpse at potentially-observable evidence for string theory, and an advancement of our understanding of information as a fundamental unit of nature.

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