C^k-regular extremal Reissner-Nördstrom black holes in AdS_4 and the third law of black hole thermodynamics

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ABSTRACT

In this work we construct examples of black hole formation from regular, one-ended asymptotically flat Cauchy data for the Einstein–Maxwell-charged scalar field system in AdS^3 spherical symmetry which are exactly isometric to the AdS^3 Schwarzschild Black Holes after a finite advanced time along the event horizon. Furthermore, the apparent horizon of said Cauchy Data coincides with that of vacuum AdS_3 at finite earlier advanced time.

This paper begins with a brief introduction to the history of black hole thermodynamics and the prerequisite Einstein-Maxwell theory required for our analysis (adapted to $\Lambda \neq 0$). However, the focus of this work is twofold: we first examine the novel C^k gluing techniques pioneered by Stefanos Aretakis [Are21], and applied to extremal black holes by Cristoph Kehle and Ryan Unger [Ung22]; adapting them to a curved spacetime.

We begin with an introduction to the prerequisite Einstein-Maxwell theory and the literature on manifold gluing theory, tracing the the lineage of the third law of black hole thermodynamics up to Kehle & Unger's 2022 work, the basepoint for our extension to curved spacetime. We adapt the machinery from [Ung22] to Schwarzschild and Reissner-Nördstrom solutions in curved spacetime. Then, we study the relevant manifold gluing theory and reprove necessary gluing theorems in dS_4 & AdS_4. Finally, we study the violation of the third law of black hole thermodynamics in AdS_4 spacetime.

1. INTRODUCTION

The study of black holes as a consequence of *gravitational collapse*, that is a black hole spacetime containing a one-ended Cauchy surface which lies outside of the black hole region, was constructed by Oppenheimer and Snyder [OS39] for the Einstein-massive dust model in spherical symmetry. Naturally, this model was the theoretical framework from which black holes were first studied as thermodynamic objects, where Bardeen, Carter, and Hawking began the analogy of black holes to classical thermodynamics with their proposal of the *four laws of black hole thermodynamics* [BCH73]. In particular, letting the surface gravity κ of the black hole take the role of its temperature, they proposed a third law in analogy to "Nernst's theorem" in classical thermodynamics.

The proposed third law of black hole thermodynamics [BCH73]: *A subextremal black hole cannot become extremal in finite time by any continuous process, no matter how idealized, in which the spacetime and matter fields remain regular and obey the weak energy condition.*

This is the origin of the long-standing analogy between black hole thermodynamics and their classical counterparts. It was also served as the groundwork for W. Israel's work on proving the third law of black hole thermodynamics (see: [Isr86; Isr92]), which he, along with others who narrowed the scope of the third law, established as follows:

The third law of black hole thermodynamics [Isr92] *any continuous process in which the stress-energy tensor of the accreted matter stays bounded and satisfies the weak energy condition in a neighborhood of the outer apparent horizon.*

However, Israel's proof had made an incorrect assumption about the nature of apparent horizons. In the context of [Isr92] and [Ung22], the apparent horizon the set of points in a given spacetime such that all emanating null geodesics converge. In the static case, this corresponds to the definition of an event horizon, however it's dynamical, local description of convergence is more suitable for the dynamical black holes for which the third law applies.

This rigorous definition of the third law of black hole thermodynamics was shown to be incorrect by Kehle and Unger, who utilized manifold gluing techniques to achieve extremality under Israel's definition, achieving C^k regular solutions from regular, one-ended asymptotically flat Cauchy data. The work done by Kehle and Unger is manifestly a disproof of Israel's statement of the third law of black hole thernodymanics embdedded in a more general theorem regarding the existence initial regular Cauchy data which evolves into an extremal black hole in finite advanced time.

Their proof relies on the gluing of two different C^k Cauchy data sets along a null cone to glue flat $\mathbb{R}^{3+1}$ Minkowski space to the Schwarzschild solution (Corollary 2.1) and later the an arbitrary Reissner-Nördstrom solution (Corollary 2.2) in finite advanced time. This allows them to product extremal black holes for specific choice of initial and final data sets, but they restrict their work to the case of $\Lambda = 0$. In this work we extend much of the theory to de Sitter and Anti-de Sitter spacetime, particularly extneding the gluing theorems to both and extremality to AdS space.

2. PREREQUISITE EINSTEIN-MAXWELL THEORY

To make meaningful claims regarding the evolution of black holes, we ought to be consice in our definitions of terms. We furthermore will reference [Ung22] repeatedly throughout this section, as the groundwork for any new theory in AdS_3 will be nearly identical to the $\Lambda = 0$ case described by Kehle and Unger.

We will consider the following Einstein-Maxwell-charged scalar field system as the equations whose initial conditions will be the Cauchy data:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 2(T^{EM}_{\mu\nu} + T^{CSF}_{\mu\nu}) \tag{1}$$

$$\nabla_\mu F_{\mu\nu} = 2q\Im(\phi\overline{D_\nu\phi})g^{\mu\nu}D_\mu D_\nu\phi = 0 \tag{2}$$

$$g^{\mu\nu}D_\mu D_\nu \phi = 0 \tag{3}$$

Where we define $T^{EM}_{\mu\nu}$ and $T^{CSF}_{\mu\nu}$ in the standard form:

$$T^{EM}_{\mu\nu} = g^{\alpha\beta}F_{\alpha\nu}F_{\beta\mu} - \frac{1}{4}F^{\alpha\beta}F_{\alpha\beta}g_{\mu\nu} \tag{4}$$

$$T^{CSF}_{\mu\nu} = Re(D_\mu\phi\overline{D_\nu\phi}) - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}D_\alpha\phi\overline{D_\beta\phi} \tag{5}$$

for a quintuplet $(\mathcal{M}, g, F, A, \phi)$[1], where $(\mathcal{M}, g)$ is a 3+1 dimensional Lorentzian manifold, ϕ is a complex scalar field, A is a real one-form, $F = dA$ is a real-valued 2-form, and $D = d + iqA$ is the gauge covariant derivative, $q \in \mathbb{R}/\{0\}$ is a fixed coupling constant for the scalar field ϕ.

We assume F is $SO(3)$-symmetric, therefore we reduce it to the simpler form:

$$F = F_{uv}du \wedge dv + F_{\theta\psi}d\theta \wedge d\psi \tag{6}$$

where $F_{\theta\psi} = Q_m sin(\theta)$ for some constant Q_m to account for the spherical choice of coordinates. We furthermore have the constraint of $U(1)$ gauge invariance of the scalar field, and thus have invariance under the local transformation: $\phi \to e^{-qi\xi}\phi$ and $A_\mu \to A_\mu + \partial_\mu\xi$ for any smooth choice of ξ. We break this symmetry by choosing ξ such that $A_v = 0$, which will be the case for any argument in this paper as well as any reference to [Ung22] or [Van20].

[1] The next few sections assume $\Lambda = 0$, as these sections pertain to background leading up to the main theorems of [Ung22].

3. PREREQUISITE MANIFOLD GLUING THEORY

Due to the nature of the gluing and the extent to which we will use the theory, it is necessary to understand the theory pioneered by Aretakis, Czimek, and Rodnianski and their work on C^k characteristic gluing of codimension-10 surfaces along null geodesics.

The theory diverges from previous work on constructing solutions along spacelike surfaces subject to traditional elliptic constraint equations [Are21], and instead considers a Lorentzian metric Σ and the problem of *characteristic gluing*. This allowed [Are21] to study the adaptability of given solutions for a given set of data posed on an initial null hypersurface. This key step relies the following setup: Consider a null hypersurface ($\mathcal{M}$) foliated by 2-dimensional compact manifolds, each diffeomorphic to S_2. Constructing the analogous constraint equations now for $\mathcal{M}$ gives a set of null constraint equations (See: [Are21])

These equations differ from the previous restriction equations for spacelike gluing in that they are heavily degenerate, and can be analyzed by parameterizing the conformal class of g for a degenerate double-null metric g on $\mathcal{M}$. This parameterization gives rise to a system of "linear control" on the solution set between any two arbitrary data sets S_1, S_2 defined on compact 2-manifolds which foliate Σ. By the ODE nature of the constraint equations, these gluing constructions result in any solution between S_1 and S_2 being the *unique* solution.

For the range of our application of the gluing methods employed by [Are21], we consider the following setup: Let $(\mathcal{M}_1, g_1)$, $(\mathcal{M}_2, g_2)$ be vacuum spacetimes with S_1, S_2 be elements of the respective foliation of spacetimes $\mathcal{M}_1, \mathcal{M}_2$.[2] We then define the *sphere perturbation* of D_2 defined on S_2 as follows: Consider an ingoing null hypersurface $\mathcal{H}_2$ which intersects S_2, and consider the restriction of the Cauchy data of $\mathcal{H}_2$ onto S_2 as D_2', the sphere perturbation of D_2 on S_2 in $\mathcal{M}_2$. [Are21] also defines a *sphere diffeomorphism* as the pullback $\phi_*(D_2)$ for some diffeomorphism ϕ of S_2.

Then, the primary groundwork of [Ung22] can be understood from the following theorem:

Theorem 3.1 (Aretakis, Czimek, Rodnianski) Let $\delta > 0$ be a real number. Consider sphere data D_1 on a sphere S_1, and characteristic initial data $x_{[-\delta,\delta],2}$ along an ingoing null hypersurface H_2, and let S_2 be a section of $\mathcal{H}_2$ with sphere data D_2 (see Sections 2.1, 2.3 and 2.4). Assume that both D_1 and $x_{[-\delta,\delta],2}$ are respectively sufficiently close to the sphere data on a sphere of radius 1 and characteristic initial data on the ingoing null hypersurface $\mathcal{H}_2$ passing through the sphere of radius 2 in Minkowski. Then there is a null hypersurface $\mathcal{H}'_{[1,2]}$ connecting the sphere data D_1 on S_1 to a perturbation $S_2' \subset \mathcal{H}_2$ of the sphere S_2 with sphere data D_2' satisfying the null constraint equations such that – up to the 10 gauge-invariant charges explicitly defined at S_2': all derivatives tangent to $\mathcal{H}'_{[1,2]}$ for sphere data D_1, D_2' are glued.

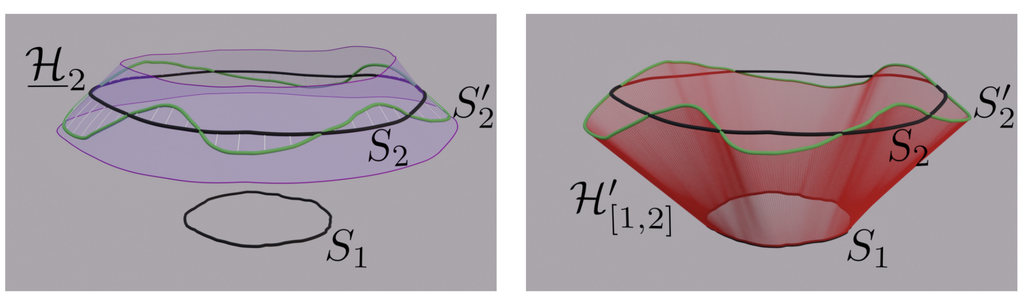

photo courtesy of [Are21]

This sufficient closeness to the sphere data in Minkowski space reduces the characteristic gluing problem to a linearized characteristic gluing in Minkowski up to diffeomorphism, where the quantities which must be glued can be matched by a choice of linearized free data along the gluing null hypersurface.

[2] We also consider all references to sphere data and/or characteristic gluing to be defined as in [Ung22] referenced below, and use double null coordinates for all g_i.

4. ARETAKIS GLUING UNDER EINSTEIN-MAXWELL SCALAR FIELD EQUATIONS

Due to the nature of this paper's existence as an extension of the gluing theorems developed in [Ung22], we first examine the theory in Minkowski as developed by Kehle and Unger.

Below is the main proposition from this work. Since this proof is a consequence of standard ODE theory, we do not repeat the proof of the main proposition here. Instead, we suggest the reader reference [Ung22] for a contextualization of the argument.

We first need to outline some basic terminology. Our assumption of spherical symmetry allows us to write g as

$$g = g_{\mathcal{Q}} + r^2 g_{S^2} \tag{7}$$

$$g = -\Omega^2 du dv + r^2 g_{S^2} \tag{8}$$

where $(\mathcal{Q}, g_{\mathcal{Q}})$ is a (1+1)-dimensional Lorentzian spacetime with (possibly empty) boundary Γ, g_{S^2} is the round metric on the sphere, and $r : \mathcal{Q} \to \mathbb{R}_{\geq 0}$ is the area-radius function. This foliates the spacetime into spheres with a center Γ which is fixed under $SO(3)$.

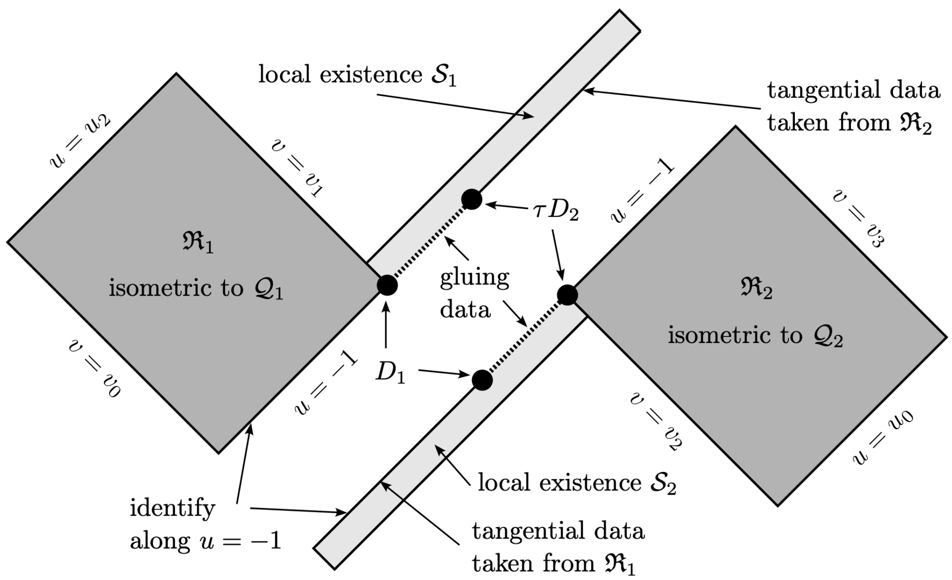

photo courtesy of [Ung22]

Definition 4.1 *(Kehle & Unger): Let $k \in \mathbb{N}$. Let $D_1, D_2 \in D_k$ be sphere data sets*[3]*. We say that D_1 can be characteristically glued to D_2 to order k in the Einstein–Maxwell-charged scalar field system in spherical symmetry if there exist $v_1 < v_2$ and a C_k cone data set $D : [v_1, v_2] \to D_k$ such that $D(v_1)$ is gauge equivalent to D_1 and $D(v_2)$ is gauge equivalent to D_2*

Proposition 4.1 *(Kehle & Unger): Let $(\mathcal{Q}_1, r_1, \Omega_1^2, \phi_1, Q_1, A_{u1})$ and $(\mathcal{Q}_2, r_2, \Omega_2^2, \phi_2, Q_2, A_{u2})$ be two C^k solutions of the EMSCF system under $SO(3)$ symmetry where each $\mathcal{Q}_i$ denotes a double null rectangle, i.e.,*

$$\mathcal{Q}_i = [u_{0,i}, u_{1,i}] \times [v_{0,i}, v_{1,i}] \tag{9}$$

Let D_1 be the sphere data induced by the first solution on $(u_{0,1}, v_{1,1})$ and D_2 be the sphere data induced by the second solution on $(u_{1,2}, v_{0,2})$. If D_1 can be characteristically glued to D_2 to order k, then there exists a spherically symmetric C^k solution $(\mathcal{Q}, r, \Omega^2, \phi, Q, A_u)$ of the EMSCF system with the following property: There exists a double null gauge (u, v) on $\mathcal{Q}$ containing double null rectangles

[3] For an understanding of Sphere data sets, consider section (5)

$$\mathcal{R}_1 = [-1, u_2] \times [v_0, v_1] \tag{10}$$
$$\mathcal{R}_2 = [u_0, -1] \times [v_2, v_3] \tag{11}$$

such that the restricted solutions $(\mathcal{R}_i, r, \Omega^2, \phi, Q, A_u)$ are isometric to $(\mathcal{Q}_i, r_i, \Omega_i^2, \phi_i, Q_i, A_{ui})$ for $i \in \{1, 2\}$, and the sphere data induced on $(-1, v_1)$ and $(-1, v_2)$ equal D_1, D_2 to kth order, respectively.

5. SPHERICALLY SYMMETRIC EMSCF EQUATIONS WITH $\Lambda \neq 0$

To examine how a non-Minkowski background spacetime will affect the formation of extremal black holes, we must first establish a system of Einstein-Maxwell Scalar field equations in curved space. Note that we still have the same two parallel transport equations from the condition that

$$\nabla^\mu F_{\nu\mu} = 0 \tag{12}$$

and thus the Maxwell equations remain unchanged (see [Kom13] for details):

$$\partial_\mu Q = -qr^2 \Im(\phi \overline{D_\mu \phi}) \partial_\nu Q = qr^2 \Im(\phi \overline{D_\nu \phi}) \tag{13}$$

Furthermore, we retain the gauge choice $A_u = 0$, and thus have

$$F = \frac{Q\Omega^2}{2r^2} du \wedge dv \partial_v A_u = -\frac{Q\Omega^2}{2r^2} \tag{14}$$

For a derivation of equations analogous to those in [Ung22][4], we reference [Ros23] appendix A. Note the following form of the wave equations (where we set $m = 0$ from [Ros23], which will be referenced in further chapters.

$$\partial_u \partial_v \phi = -\frac{\partial_u \phi \partial_v r}{r} - \frac{\partial_u r \partial_v \phi}{r} + \frac{iq\Omega^2 Q}{4r^2}\phi - iqA_u \frac{\partial_v r}{r}\phi - iqA_u \partial_v \phi \tag{15}$$
$$\partial_u \partial_v r = \frac{\Omega^2}{4r^3}Q^2 + \frac{\Omega^2 r}{4}\Lambda - \frac{\Omega^2}{4r} - \frac{\partial_v r \partial_u r}{r} \tag{16}$$
$$\partial_u \partial_v log(\Omega^2) = -2Re(D_u \phi \overline{\partial_v \phi}) - \frac{\Omega^2 Q^2}{r^4} + \frac{\Omega^2}{2r^2} + \frac{2\partial_v r \partial_u r}{r^2} \tag{17}$$

Where equation (2.10) is modified slightly to include a nonzero contribution from A_u as defined in equation (2.4) of [Ung22]. We also have an identical set of Rayshaudhuri's equations as in the Minkowski case:

$$\partial_u\left(\frac{\partial_u r}{\Omega^2}\right) = -r\frac{|D_u \phi|^2}{\Omega^2} \tag{18}$$
$$\partial_v\left(\frac{\partial_v r}{\Omega^2}\right) = -r\frac{|D_v \phi|^2}{\Omega^2} \tag{19}$$

6. FORMULATION OF PREREQUISITE GLUING PROPOSITIONS

Before we reach the main gluing theorems, we must first define C^k-regularity as well as how we prescribe initial Cauchy data in a curved spacetime. This section will mirror sections 2.2 and 2.3 of [Ung22], with only minor changes to adapt proofs to a curved backgound spacetime.

Definition 6.1 *(Kehle & Unger): Let $k \in \mathbb{N}$. A C^k solution for the EMCSF system in the gauge $A_u = 0$ will consist of a codomain $\mathcal{Q} \subset \mathbb{R}^{1+1}_{u,v}$ as well as functions $r \in C^{k+1}(\mathcal{Q})$ and $(\Omega^2, \phi, Q, A_u \in C^k(\mathcal{Q})$ such that $r > 0$, $\Omega^2 > 0$[5], ϕ has complex codomain, $\partial_v^{k+1} A_u \in C^0(\mathcal{Q})$, and the functions satisfy the given EMSCF system.*

[4] Equations (2.4)-(2.11)

[5] we will fix this to be 1 in the double null gauge

To prescribe initial Cauchy data, we first need a Cauchy surface onto which we can define our C^k solution. We define the *bifurcate null hypersurface* $C \cup \underline{C} \subset \mathbb{R}^{1+1}_{u,v}$ to be

$$C = \{u = -1\} \cap \{v \geq 0\} \underline{C} = \{v = 0\} \cap \{u \geq -1\} \tag{20}$$

Where each point in $C \cup \underline{C}$ corresponds to a foliation diffeomorphic to S^2. The point which defines $C \cap \underline{C}$ is referred to as the *bifurcation sphere*. We pose initial data on $C \cup \underline{C}$. We define such a data set as a C^k *bifurcate characteristic initial data set*, which corresponds to a pentupluet of functions $(\Omega^2, \phi, Q, A_u \in C^k(\mathcal{Q})$ defined on the bifurcate null hypersurface such that they satisfy the EMSCF equations as well as Definition 2.1.

Proposition 2.1 from [Ung22] remains unchanged as well, as the guaranteed existence of a solution in a local neighborhood is a consequence of a broader ODE theory, and thus is invariant under a change in Λ. We follow a standard iteration argument to show a relevant proposition:

Proposition 6.2 *(Kehle & Unger): Let $(r, \Omega^2, \phi, Q, A_u)$ be a C^k bifurcate characteristic initial data set as in Definition 2.2. Then the EMSCF system can be used to determine as many as u-derivatives of $r, \Omega^2, \phi, Q,$ and A_u on C as is consistent with Definition 2.1 explicitly from the data on $C \cup \underline{C}$*

Since $(r, \Omega^2, \phi, Q, A_u)$ are given on $\underline{C}$, ∂_u^i is well-defined for $1 \leq i \leq k$[6] on the characteristic data. Thus $\partial_u r(-1, 0)$ is known. Rewriting the wave equation for r, we have

$$(\partial_u + \frac{\partial_v r}{r})\partial_u r = \frac{\Omega^2}{4r^3}Q^2 + \frac{\Omega^2 r}{4}\Lambda - \frac{\Omega^2}{4r} \tag{21}$$

where, as in the case of $\Lambda = 0$, the right-hand side terms are all known and therefore by standard ODE theory we know $\partial_u r(-1, v)$. Following this strategy, $\partial_u \phi(-1, v)$, $\partial_u log(\Omega^2)(-1, v)$ can be found, as can $\partial_u A_u(-1, v)$. The iteration argument proceeds identically to [Ung22] from here, giving us k derivatives for $r, \Omega^2, \phi, Q)$, and $k+1$ derivatives for A_u.

We now setup a notion of initial sphere data. This definition is indistinguishable to that of Aretakis, Czimek, Rodnianski.

Definition 6.2*: A sphere data set with regularity index k for the EMSCF system of equations (with $A_v = 0$) is the following set:*

$$\rho > 0, \rho_u^1, ..., \rho_u^{k+1}, \rho_v^1, ..., \rho_v^{k+1} \in \mathbb{R} \tag{22}$$
$$\omega > 0, \omega_u^1, ..., \omega_u^k, \omega_v^1, ...\omega_v^k \in \mathbb{R} \tag{23}$$
$$\phi, \phi_u^1, ..., \phi_u^k, ..., \phi_v^1, ..., \phi_v^k \in \mathbb{C} \tag{24}$$
$$\zeta, \zeta u^1, ..., \zeta u^k, \zeta v^1, ..., \zeta v^k \in \mathbb{R} \tag{25}$$
$$a, a_u^1, ..., a_u^k, a_v^1, ..., a_v^k, a_v^{k+1} \in \mathbb{R} \tag{26}$$
$$\tag{27}$$

As in [Ung22], denote $\mathcal{D}_k$ to be the set of sphere data sets with regularity index k, where the sphere data sets are degenerate with respect to the solutions they generate under the following gauge group:

Definition 6.3 *(Kehle & Unger): The full gauge group of the Einstein-Maxwell charged scalar field system in spherically symmetric double null gauge with $A_v = 0$ is formalized as:*

$$\mathcal{G} = \{(f, g) : f, g \in \text{Diff}_+(\mathbb{R}), f(0) = g(0) = 0\} \times C^\infty(\mathbb{R})$$

[6] C^{k+1} regularity for A_u

We will also assume the gauge $\Omega = 1$; that is $\omega = 1$, $\omega_u^i = \omega_v^i = 0$ for $1 \leq i \leq k$. Every sphere data set then is gauge equivalent to a lapse normalized sphere data set. (Defined in *Kehle & Unger Definition 2.5*)

Becuase the proofs of Propositions 2.3 and 2.4 from [Ung22] do not have proofs which involve any machinery affected by a nonzero Λ, we leave the reader to reference [Ung22] for the proofs and state only the propositions themselves, along with the relevant definitions). We restate them here for the sake of continuity:

Proposition 6.3 *(Kehle & Unger): Let $k \in \mathbb{N}$, $v_1 < v_2 \in \mathbb{R}$, $r, A_u \in C^{k+1}([v_1, v_2]$, and $\Omega^{@}, \phi, Q \in C^k([v_1, v_2])$ which satisfy the EMSCF Maxwell and Raychaudhuri equations on $[v_1, v_2]$. Let $D_1 \in D_k$ such that all v-components of D_1 agree with the corresponding v-derivatives of $(r, \Omega^2, \phi, Q, A_u)$ at v_1. Then there exists a unique continuous function $D : [v_1, v_2] \to D_k$ such that $D(v_1) = D_1$ and upon formal identification of the formal symbols $\rho(D_v), \rho_v^1(D(v))$, etc, with $(r, \Omega^2, \phi, Q, A_u)$ at v_1 and their u, v derivatives, satisfies the EMSCF system and agrees with $(r, \Omega^2, \phi, Q, A_u)$ at v_1 in v-components for all $v \in [v_1, v_2]$.*

Definition 6.4 *(Kehle & Unger): Let $k \in \mathbb{N}$, $v_1 < v_2 \in \mathbb{R}$. A C^k* ***cone data set*** *for teh EMSCF scalar field in symmetry is a continuous function $D : [v_1, v_2] \to D_k$ satisfying the EMSCF system.*

Proposition 6.4 *(Kehle & Unger): Let $k \in \mathbb{N}$, $v_1 < v_2 \in \mathbb{R}$, and $D_1 \in D_k$ be lapsed normalized. For any $\phi \in C^k([v_1, v_2])$ such that $\partial_v^i \phi(D_1)$, for $0 \leq i \leq k$, there exist unique functions $r, A_u \in C^{k+!}([v_1, v_2])$ and $Q \in C^k([v_1, v_2])$ such that $(r, \Omega^2, \phi, Q, A_u)$ satisfies the hypotheiss of Proposition 2.3 with $\Omega^2(v) = 1$ for every $v \in [v_1, v_2]$.*

7. SPHERE DATA SETS IN CURVED SPACETIME

Here we define canonical sphere data sets in dS, AdS space. In our characterization of initial sphere data sets we wish to use as the endpoints for our null gluing light cone, we limit the scope to spherically-symmetric solutions of the EMSCF system. Formally, we define spherical symmetry as follows:

Definition 7.1*: Let $(\mathcal{M}, g, F, A, \phi)$ be smooth data set satisfying the EMSCF system. Such data is spherically symmetric if:*

1. *The Lie group SO(3) acts by isometry on $(\mathcal{M}, g)$*
2. *The SO(3) action preserves g, ϕ, $|\phi|^2, A$*
3. *$\Sigma = \mathcal{M}/SO(3)$ is a connected 1-dimensional Riemannian manifold*

We now define sphere data sets for dS, AdS space. We use the standard metric shown below for the definition of the following sets. We associate the metric:

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)^2}dr^2 + r^2 g_{S^2} \tag{28}$$

$$f(r) = 1 \mp \frac{\Lambda r^2}{3} - \frac{2M}{r} \tag{29}$$

for $\pm$ corresponding to De Sitter and Anti-de Sitter space, respectively.

Definition 7.2*: Let $(\mathcal{M}, g, F, A, \phi)$ be smooth data set satisfying the EMSCF system. Such data is spherically symmetric if:*

1. *The Lie group SO(3) acts by isometry on $(\mathcal{M}, g)$*
2. *The SO(3) action preserves g, ϕ, $|\phi|^2, A$*
3. *$\Sigma = \mathcal{M}/SO(3)$ is a connected 1-dimensional Riemannian manifold*

Definition 7.3*: de Sitter data. Let $k \in \mathbb{N}$, and $R > 0$. Then the unique lapse normalize sphere data set satisfying*

1. *$\rho = R$*

2. $\rho_u^1 = -\frac{1}{2}(1 - \frac{1}{3}\Lambda R^2)$
3. $\rho_v^1 = \frac{1}{2}(1 - \frac{1}{3}\Lambda R^2)$

is herein referred to as the dS sphere data of radius R and vacuum energy $\Lambda > 0$. It is notated $D_{dS}^{R,k}$.[7]

Definition 7.4: *Anti de-Sitter data. Let $k \in \mathbb{N}$, and $R > 0$. Then the unique lapse normalize sphere data set satisfying*

1. $\rho = R$
2. $\rho_u^1 = -\frac{1}{2}(1 - \frac{1}{3}\Lambda R^2)$
3. $\rho_v^1 = \frac{1}{2}(1 - \frac{1}{3}\Lambda R^2)$

is the vacuum Anti de-Sitter sphere data set of mass M and cosmological constant $\Lambda < 0$, notated $D_{AdS}^{R,k,M}$

Definition 7.5: *de Sitter-Schwarzschild data. Let $k \in \mathbb{N}$, and $0 \leq r_{sch} \leq R \leq r_{ch}$, where r_{sch} is the dS-Schwarzschild radius and r_{ch} is the Cosmological horizon. We assume to work in a regime for $M >> |\Lambda|$, such that $0 < r_{sch} < r_{ch}$. Then the unique lapse normalize sphere data set satisfying*

1. $\rho = R$
2. $\rho_u^1 = -\frac{1}{2}(1 - \frac{1}{3}\Lambda R^2)$
3. $\rho_v^1 = \frac{1}{2}(1 - \frac{2M}{R} - \frac{1}{3}\Lambda R^2)$

is the Schwarzschild-de Sitter sphere data set of mass M and cosmological constant $\Lambda > 0$, notated $D_{SdS}^{R,k,M}$

Definition 7.6: *AdS-Schwarzschild data. Let $k \in \mathbb{N}$, and $0 \leq r'_{sch} \leq R$, where r'_{sch} is the apparent horizon of a SAdS black hole with parameters Λ, M. Then the unique lapse normalize sphere data set satisfying*

1. $\rho = R$
2. $\rho_u^1 = -\frac{1}{2}(1 - \frac{1}{3}\Lambda R^2)$
3. $\rho_v^1 = \frac{1}{2}(1 - \frac{2M}{R} - \frac{1}{3}\Lambda R^2)$

is the Schwarzschild Anti-de Sitter sphere data set of mass M and cosmological constant $\Lambda < 0$, notated $D_{SAdS}^{R,k,M}$

Definition 7.7: *RNAdS data. Let $k \in \mathbb{N}$, and $1 - \frac{2M}{r} + \frac{e^2}{r^2} - \frac{\Lambda r^2}{3}$ with two roots, $r_+, r_- > 0$ corresponding to the inner and outer horizons of the black hole. Then the unique lapse normalize sphere data set satisfying*

1. $\rho = r_+$
2. $\rho_u^1 = -1/2$
3. $\rho_v^1 = 0$
4. $q = e$

is the Reissner-Nördstrom Anti-de Sitter sphere data set of mass M and cosmological constant Λ, notated $D_{RNAdS}^{R,k,M}$

Lemma 7.1: *Gauge Transformation dS/AdS identificaiton. If $D \in D_k$ satisfies (for all $1 \leq i \leq k$):*

1. $\rho = R > 0$
2. $\rho_u^1 < 0$
3. $\frac{1}{2}\rho(1 + 4\rho_u^{|}1\rho_v^1) = M$

[7] For any sphere data set, all values listed are set to zero.

4. $Q = 0$

5. $\zeta^i_u = \zeta^i_v = 0$

Then it follows that R is equivalent to $D^{R,k,M}_{SdS}/D^{R,k,M}_{SAdS}$ under $\mathcal{G}$.

Proof: First, note that from Maxwell's equations $q = 0 \to \partial^i_{u,v} q = 0$ for any derivatives of u, v to all orders. Additionally, since $\zeta^i_u = \zeta^i_v = 0$, we can apply a gauge transformation χ such that $d\chi = -1$, then $0 = \phi \to e^{-ie\chi}\phi = 0$ and $A \to A + d\chi = 0$ and thus $a^i_u = 0$ for $0 \leq i \leq k$. $a^i_v = 0$ for $1 \leq i \leq k$ from $F = d(A_u du)$. Then rescale the lapse such that $\omega = 1$. Thus all assumptions are satisfied, as $R > 2M$ follows from $\rho^1_u \rho^1_v < 0$.

Similarly, we define gauge equivalence for $D^{R,k,M}_{RNAdS}$:

Lemma 7.2: *Gauge Transformation for RNAdS: If $D \in D_k$ satisfies (for all $1 \leq i \leq k$):*

1. $\rho = r_+$

2. $\rho^1_u < 0$

3. $\rho^1_v = 0$

4. $e = qM$

5. $\zeta^i_u = \zeta^i_v = 0$

The charge vanishes to all orders, so under a normalization of gauge potential and lapse we rescale $u \to \lambda u$ and $v \to \lambda^{-1} v$ to make $\rho^1_u = -\frac{1}{2}$.

8. K-REGULAR GLUING THEOREMS IN CURVED SPACETIME

This section will trace the proofs in Section 3.4 of [Ung22]. We first show the cases of gluing $D^{R,k,M}_{dS}$, $D^{R,k,M}_{AdS}$ to dS/AdS-Schwarzschild Cauchy data sphere sets $D^{R,k,M}_{SdS}$, $D^{R,k,M}_{SAdS}$ respectively.

Theorem 8.1: *For any $k \in \mathbb{N}$, $0 < R < 2M$, and $\Lambda << r$,*[8] *the de Sitter sphere of radius R, $D^{R,k,M}_{dS}$, can be characteristically glued to a Schwarzschild-de Sitter event horizon sphere with mass M, $D^{R,k,M}_{SdS}$ with C^k regularity, such that the solution globally satisfies the EMSCF system with SO(3) symmetry.*

Theorem 8.2: *For any $k \in \mathbb{N}$ and $0 < R < 2M$, the Anti-de Sitter sphere of radius R, $D^{R,k,M}_{AdS}$, can be characteristically glued to an Anti-de Sitter Schwarzschild event horizon sphere with mass M, $D^{R,k,M}_{ASdS}$ with C^k regularity, such that the solution globally satisfies the EMSCF system with SO(3) symmetry.*

Theorem 8.3: *For any $k \in \mathbb{N}$, $\mathfrak{q} \in [-1, 1]$, and $e \in \mathbb{R} \setminus \{0\}$, there exists a number $M_0(k, \mathfrak{q}, e) \geq 0$ such that if $M_f > M_0$, $0 \leq M_i \leq \frac{1}{8} M_f$, and $2M_i < R_i \leq \frac{1}{2} M_f$, then the Anti-de Sitter Schwarzschild sphere of mass M_i and radius R_i, $D^{\mathrm{S}}_{M_i,R_i,k}$, can be characteristically glued to the Reissner-Nördstrom event horizon with mass M_f and charge to mass ratio q, $D^{R,k,M}_{RNAdS}$, to order C^k. Furthermore, associated characteristic data can be chosen to have no spherically symmetric antitrapped surfaces.*[9]

9. PROOF OF THEOREMS 8.1 & 8.2

We begin by making the following ansatz: We claim that the scalar field can be constructed as a collection of analytic "pulses", each with compact support. Formally, we partition the unit line into k sections, with

$$0 = v_0 < v_1 < ... < v_k < v_{k+1} = 1 \tag{30}$$

such that for each $1 \leq j \leq k+1$, we fix a nontrivial bump function

[8] See: dS gluing limit; Appendix A

[9] reduces to "Theorem 2B" [Ung22] in the case of $\Lambda = 0$

$$\chi_j \in C_c^\infty((v_{j-1}, v_j); \mathbb{R}) \tag{31}$$

We then define the scalar field ansatz as follows: Let $\alpha \in \mathbb{R}^{k+1}$, and define

$$\phi_\alpha(v) = \Sigma_{j=1}^{k+1} \alpha_j \chi_j(v) \tag{32}$$

Where α_j is the jth component of α.

Using lapse-normalized equivalence, we set $\Omega^2 \phi_\alpha(v) = 1$ on $v \in [0,1]$. From equation (18), lapse-normalization, along with the scalar field ansatz, determines a unique $r_\alpha(v)$ along $[0,1]$:

$$\partial_v^2 r_\alpha(v) = -r_\alpha(v)(\partial_v \phi_\alpha(v))^2 \tag{33}$$

We then uniquely fix the solution by assigning values to $r_\alpha(v)$ at $v = 1$:

$$r_\alpha(1) = 2M \tag{34}$$

$$\partial_v r_\alpha(1) = 0 \tag{35}$$

Let $0 < \epsilon < 2M - R$. By Cauchy Stability and monotonicity properties of equation (18), there exists some $\delta > 0$ such that for all $0 \leq |\alpha| < \delta$,

$$sup_{[0,1]} |r_\alpha(v) - 2M| \leq \epsilon \tag{36}$$

$$inf_{[0,1]} \partial_v r_\alpha(v) \geq 0 \tag{37}$$

$$\partial_v r_\alpha(0) > 0 \tag{38}$$

We now consider the restriction of α to the sphere $S_\delta^k = \{\alpha \in \mathbb{R}^{k+1}; |\alpha| = \delta\}$. For all such α, define $D_\alpha(0) \in D_k$ as the sphere data set satisfying:

$$\rho = r_\alpha(0) > 0$$
$$\rho_v^1 = \partial_v r_\alpha(0) > 0$$
$$\rho_u^1 = \frac{-1}{4\rho_v^1}$$
$$\omega = 1$$

By Lemma 7.1, $D_\alpha(0)$ is equivalent to $D_{dS}^{R,k,M}$ under the gauge group $\mathcal{G}$. For each $\alpha \in S_\delta^k$, we can apply Propositions 6.3 & 6.4 to uniquely determine sphere data along the null cone parameterized by $v \in [0,1]$:

$$D_\alpha : [0,1] \to \mathcal{D}_k \tag{39}$$

where $D_\alpha(0)$ is given above. By standard ODE theory, $D_\alpha(v)$ is continuous in both v and α. Recall the notation for Cauchy sphere data sets established above; $\rho(D_\alpha(v)) = r_\alpha(v)$, $\zeta(D_\alpha(v)) = \phi_\alpha(v)$, and $\partial_u^i \phi_\alpha(v) = \zeta_u^i(D_\alpha(v))$.
By construction, $D_\alpha(1)$ satisfies

1. $\rho = 2M$
2. $\rho_u^1 < 0$
3. $\rho_v^1 = 0$
4. $\omega = 1$
5. $\zeta_v^i = 0$ for $0 \leq i \leq k$.

Where conditions 1 and 3 follow from the conditions set on $r_\alpha(v)$ above, condition 5 follows from the compact support of ϕ, and condition 4 is lapse normalization. Condition 2 can be shown by considering the quantity $\partial_v(r\partial_u r)$, which can be shown via expanding and substitution of (16) to be:

$$\partial_v(r\partial_u r) = -\frac{1}{4}(1 - r^2\Lambda) \tag{40}$$

which propagates the sign of $r\partial_u(r)$ for small values of Λ, and all values of $\Lambda < 0$. See Appendix A for a specific discussion of this limit. By assumption in Theorem 8.1, this holds and therefore $\partial_u r_\alpha(v) < 0$ on $[0,1]$.

In order to glue to $D_{SdS}^{R,k,M}$ with k-regular solutions, we require $\zeta_u^i(1) = 0$ for all $1 \leq i \leq k$. This is equivalent to, for some $\tilde{\alpha} \in S_\delta^k$, our ansatz $\phi_\alpha(1)$ satisfying $\partial_u^i \phi_{\tilde{\alpha}} = 0$ for $1 \leq i \leq k$. This result is in fact guaranteed by a simple application of the Borsuk-Ulam theorem [Bor33] on S_δ^k.

Borsuk-Ulam Theorem: *Let $f : S^k \to \mathbb{R}^k$ be a continuous, odd function. Then there exists $x \in S^k$ such that $f(x) = 0$.*

***Lemma 9.1**: As functions on* $[0,1] \times S_\delta^k$, *r* and Ω^2 *are all even functions of α.* ϕ is odd in α. Furthermore, all higher u, v derivatives hold the same parity as their zeroth derivative. Importantly,

$$F : S_\delta^k \to \mathbb{R}^k$$
$$\alpha \to (\zeta_u^1, ..., \zeta_u^k)$$

is continuous and odd.

Proof: Immeadiately from the definition, the scalar field ϕ is odd in α. Since r_α is determined by equation (25) which has only even powers of ϕ, then r_α must be even in α.

We then combine wave equations (15), (16) to determine the wave equation for $r\phi$. Simple rearranging yields:

$$\partial_u \partial_v (r\phi) = \phi(\frac{r\Lambda}{4} - \frac{1}{4r} - \frac{\partial_u(r)\partial_v(r)}{r}) \tag{41}$$

Since ϕ is odd in α and the expression depending on r is even, the right-hand side of this expression is odd in α. Thus it follows by inspection that $\partial_u(r\phi)$ is odd and therefore $\partial_u \phi$ is also odd. From here it follows from Proposition 6.2 that these parity rules hold for higher order derivatives as well. This can be seen in the transport equations for ingoing derivatives of r and Ω^2 (which only contain even powers of ϕ), and the analogous equations for $\partial_u^i \phi$ for $1 \leq i \leq k$ involve only odd powers of ϕ.

Thus the map $F : S_\delta^k \to \mathbb{R}^k$ is odd. By the Borsuk-Ulam Theorem, this antipodal map must contain a root; that is some $\tilde{\alpha}$ such that $F(\tilde{\alpha}) = 0$. This then determines a Cauchy sphere at $v = 1$ which is gauge equivalent to either $D_{SdS}^{R,k,M}$ or $D_{SAdS}^{R,k,M}$ with C^k regularity. Noting the extra condition on the evolution of (16), this proves Theorems 8.1 and 8.2 simultaneously.

10. PROOF OF THEOREM 8.3

Given that this proof mirrors the proof found in Kehle & Unger's work (see: Proof of Theorem 2B), this section will sketch the proof as fully detailed in [Ung22]. Lemmas where Λ has a trivial effect will be starred* and their proofs will be fundamentally identical the original work. Notation is also drawn from [Ung22].

Let

$$0 = v_0 < v_1 < \cdots < v_{2k} < v_{2k+1} = 1 \tag{42}$$

be an arbitrary partition of $[0,1]$. For each $1 \leq j \leq 2k+1$, fix a nontrivial bump function

$$\chi_j \in C_c^\infty((v_{j-1}, v_j); \mathbb{R}).$$

In the rest of this section, the functions $\chi_1, \ldots, \chi_{2k+1}$ are fixed and our constructions depend on these choices.[10]

For $\alpha = (\alpha_1, \ldots, \alpha_{2k+1}) \in \mathbb{R}^{2k+1}$, set

$$\phi_\alpha(v) \doteq \phi(v; \alpha) \doteq \sum_{1 \leq j \leq 2k+1} \alpha_j \chi_j(v) e^{-iv}. \tag{43}$$

[10] If $e > 0$, this choice of ϕ will make $Q \geq 0$, with is consistent with $\mathfrak{q} > 0$. If $\mathfrak{e} > 0$ and $\mathfrak{q} < 0$, then we replace $-iv$ in the exponential with $+iv$. Similarly, the cases $\mathfrak{e} < 0$, $\mathfrak{q} > 0$ and $\mathfrak{e} < 0$, $\mathfrak{q} < 0$ can be handled. Therefore, we assume without loss of generality that $\mathfrak{e} > 0$, $\mathfrak{q} > 0$.

For $\hat{\alpha} \in S^{2k}$, define $r(v;\beta,\hat{\alpha}) = r(v;\beta\hat{\alpha})$. Set $\Omega^2(v;\alpha) \equiv 1$. We then have:

$$\partial_v^2 r(v;\alpha) = -|\alpha|^2 r(v;\alpha)|\partial_v \phi_{\hat{\alpha}}(v)|^2, \tag{44}$$

$$\partial_v Q(v;\alpha) = e|\alpha|^2 r(v;\alpha)^2 \Im(\phi_{\hat{\alpha}}(v)\overline{\partial_v \phi_{\hat{\alpha}}(v)}). \tag{45}$$

$r_\alpha(1)$ and Q are initialized to:

$$r(1;\alpha) = r_+,$$
$$\partial_v r(1;\alpha) = 0,$$
$$Q(0;\alpha) = 0 \tag{46}$$

which, together with Maxwell's and Raychaudhuri's equations, uniquely determine r and Q on $[0,1]$.

Note the following identities:

$$|\partial_v \phi_{\hat{\alpha}}|^2 = \sum_{1\le j\le 2k+1} \hat{\alpha}_j^2 \left(\chi_j^2 + \chi_j'^2\right)$$

and

$$\Im(\phi_{\hat{\alpha}}\overline{\partial_v \phi_{\hat{\alpha}}}) = \sum_{1\le j\le 2k+1} \hat{\alpha}_j^2 \chi_j^2.$$

Therefore,

$$\int_0^1 |\partial_v \phi_{\hat{\alpha}}|^2 \, dv \approx \int_0^1 \Im(\phi_{\hat{\alpha}}\overline{\partial_v \phi_{\hat{\alpha}}}) \, dv \approx 1$$

for any $\hat{\alpha} \in S^{2k}$.

Lemma 10.4*: *There exists a constant $0 < c \lesssim 1$ such that $0 < \beta \lesssim c\hat{\alpha} \in S^{2k}$ implies that $r(\cdot;\beta\hat{\alpha})$ satisfies*

$$r(v;\beta\hat{\alpha}) \ge \tfrac{1}{2} r_+ \tag{47}$$

$$\partial_v r(v;\beta\hat{\alpha}) \ge 0 \tag{48}$$

Furthermore,

$$\partial_v r(0;\beta\hat{\alpha}) > 0. \tag{49}$$

Proof. This is a simple bootstrap argument in v. Assume that on $[v_0,1] \subset [0,1]$, we have

$$\inf_{[v_0,1]} r \ge 0 \tag{50}$$

$$\inf_{[v_0,1]} \partial_v r \ge 0. \tag{51}$$

This is clear for v_0 close to 1 by Cauchy stability. From Raychaudhuri's equation, $r \ge 0$ implies $\partial_v r$ is monotone decreasing, hence is bounded above by $\partial_v r(v_0)$, which can be estimated by

$$\partial_v r(v_0) = \int_{v_0}^1 \beta^2 r |\partial_v \phi_{\hat{\alpha}}|^2 dv \lesssim \beta^2 r_+ \tag{52}$$

since $r \le r_+$ on $[v_0,1]$. It follows that

$$r(v_0) = r_+ - \int_{v_0}^1 \partial_v r \, dv \ge r_+ - C\beta^2 r_+ \tag{53}$$

for some $C \lesssim 1$. Choosing $\beta > 0$ sufficiently small shows $r(v_0) \ge \frac{1}{2} r_+$ which improves the bootstrap assumptions and proves the first desired estimate. Finally, note the estimate $\partial_v r(0;\beta\hat{\alpha}) > 0$. holds true as $\partial_v r$ is monotone decreasing and r is not constant ($\beta > 0$ and the scalar field is not identically zero). □

Lemma 10.5: *By potentially making the constant c from w smaller, we have that for any $0 < \beta \leq c$ and $\hat{\alpha} \in S^{2k}$, the following estimate holds*

$$\frac{\partial}{\partial \beta} Q(1; \beta, \hat{\alpha}) > 0. \tag{54}$$

Proof. Integrating Maxwell's equation w and using w, we find

$$Q(1; \beta, \hat{\alpha}) = \int_0^1 e\beta^2 r^2 \Im(\phi_{\hat{\alpha}} \overline{\partial_v \phi_{\hat{\alpha}}})\, dv. \tag{55}$$

A direct computation yields

$$\partial_\beta Q(1; \beta, \hat{\alpha}) = 2e\beta \int_0^1 (r^2 + \beta r \partial_\beta r) \Im(\phi_{\hat{\alpha}} \overline{\partial_v \phi_{\hat{\alpha}}})\, dv. \tag{56}$$

Note that $\Im(\phi_{\hat{\alpha}} \overline{\partial_v \phi_{\hat{\alpha}}}) \geq 0$ pointwise and is not identically zero. Since $0 < \beta \leq c$, we estimate

$$r^2 + \beta r \partial_\beta r \geq \tfrac{1}{4} r_+^2 - C\beta r_+^2 = r_+^2 (\tfrac{1}{4} - C\beta), \tag{57}$$

where we also used $|\partial_\beta r| \lesssim r_+$ which follows directly from differentiating (44) with respect to $\beta = |\alpha|$. Therefore, by choosing c even smaller, we obtain $\partial_\beta Q(1; \beta, \hat{\alpha}) > 0$. □

Lemma 10.6* *If eM_f/q is sufficiently large depending only on k and the choice of profiles, then there is a smooth function $\beta_Q : S^{2k} \to (0, \infty)$ so that $Q(1; \beta_Q(\hat{\alpha}), \hat{\alpha}) = qM_f$ for every $\hat{\alpha} \in S^{2k}$, which also satisfies*

$$\beta_Q(\hat{\alpha}) \approx \frac{\sqrt{qM_f}}{\sqrt{e} r_+} \tag{58}$$

$$\beta_Q(-\hat{\alpha}) = \beta_Q(\hat{\alpha}) \tag{59}$$

for every $\hat{\alpha} \in S^{2k}$.

Proof. As in the proof of w we have

$$Q(1; \beta, \hat{\alpha}) = e\beta^2 \int_0^1 r^2 \Im(\phi_{\hat{\alpha}} \overline{\partial_v \phi_{\hat{\alpha}}}). \tag{60}$$

If β is sufficiently small so that w and w apply, we estimate

$$Q(1; \beta, \hat{\alpha}) \approx e\beta^2 r_+^2. \tag{61}$$

For eM_f/q sufficiently large as in the assumption, we apply now the intermediate value theorem, to obtain a $\beta_Q(\hat{\alpha})$ satisfying $0 < \beta_Q(\hat{\alpha}) \leq c$ such that

$$wQ(1; \beta_Q, \hat{\alpha}) = qM_f. \tag{62}$$

Note that $\beta_Q(\hat{\alpha})$ is unique since $Q(1; \cdot, \hat{\alpha})$ is strictly increasing as shown in w. Moreover, since $Q(1; \cdot, \cdot)$ is smooth (note that $\hat{\alpha} \in S^{2k}$ and $\beta > 0$ enter as smooth parameters in w which defines Q), a direct application of the implicit function theorem using that $\partial_\beta Q(1; \cdot, \hat{\alpha}) \neq 0$ shows that $\beta_Q : S^{2k} \to (0, \infty)$ is smooth. Moreover, β_Q satisfies

$$e\beta_Q^2 r_+^2 \approx qM_f$$

which shows our above estimate for β. Finally, note that $Q(1; \beta, -\hat{\alpha}) = Q(1; \beta, \hat{\alpha})$, from which it follows that β is odd. □

Lemma 10.7: *Let eM_f/q be sufficiently large (depending only on k and the choice of profiles) so that the previous lemma applies. Then*

$$p_Q : S^{2k} \to Q^{2k} \tag{63}$$

$$\hat{\alpha} \mapsto \beta_Q(\hat{\alpha})\hat{\alpha} \tag{64}$$

is a diffeomorphism, where

$$Q^{2k} = \{\beta_Q(\hat{\alpha})\hat{\alpha} : \hat{\alpha} \in S^{2k}\} \subset \mathbb{R}^{2k+1} \tag{65}$$

is the radial graph of β_Q. Q^{2k} is invariant under the antipodal map, and p_Q commutes with the antipodal map.

Proof. By definition of Q^{2k} and the facts that β_Q is smooth, positive, and invariant under the antipodal map as proved in w, the stated properties of Q^{2k} and p_Q follow readily. □

Having identified the set Q^{2k} which guarantees gluing of the charge Q, for the rest of the section we will always take $\alpha \in Q^{2k}$. Recall from w that for every $\alpha \in Q^{2k}$:

$$|\alpha| \approx \frac{\sqrt{qM_f}}{\sqrt{e}r_+}. \tag{66}$$

Before proceeding to choose sphere data, we will need to examine the equation for $\partial_u r$ because this will place a further restriction on α which must be taken into account before setting up the topological argument. Setting the initial hawking mass to M_i,

$$m(0;\alpha) = M_i \tag{67}$$

initializing

$$\partial_u r(0;\alpha) = -(1 - \frac{2M_i}{r(0;\alpha)})\frac{1}{4\partial_v r(0;\alpha)}. \tag{68}$$

The transverse derivative $\partial_u r(v;\alpha)$ is now determined by solving wave equation (10),

$$w\partial_v\partial_u r(v;\alpha) = -\frac{1}{4r(v;\alpha)^2} - \frac{\partial_u r(v;\alpha)\partial_v r(v;\alpha)}{r(v;\alpha)^2} + \Lambda\frac{\Omega^2 r}{4} + \frac{Q(v;\alpha)^2}{4r(v;\alpha)^3}, \tag{69}$$

with initialization w.

Furthermore,

$$1 - \frac{2M_i}{r(0;\alpha)} \geq 1 - \frac{4M_i}{M_f} > 0,$$

so

$$\partial_u r(0;\alpha) < 0. \tag{70}$$

Having initialized $\partial_u r$ at $v = 0$, we determine $\partial_u r(v;\alpha)$ using the above wave equation, and we will now show that for eM_f/q sufficiently large, $\partial_u r(v;\alpha) < 0$ for all $v \in [0,1]$.

Lemma 10.8: *If eM_f/q is sufficiently large depending only on k and the choice of profiles and if $0 \leq M_i \leq \frac{1}{8}M_f$, then*

$$\sup_{v\in[0,1]} \partial_u r(v;\alpha) < 0 \tag{71}$$

for every $\alpha \in Q^{2k}$.

Proof. Since $r > 0$ on $[0,1]$, it suffices to show that

$$\sup_{[0,1]} r\partial_u r < 0.$$

First, as in the proof of 8.1, we bound $\partial_v(r\partial_u r)$:

$$|\partial_v(r\partial_u r)| = |\frac{1}{4}(1 + \Lambda r^2 - \frac{Q^2}{r^2})| \lesssim 1, \tag{72}$$

as

$$Q(v;\alpha) \leq Q(1;\alpha) = qM_f \lesssim r(v;\alpha), \tag{73}$$

then, from Lemma 4.5,

$$\sup_{v\in[0,1]} r(v)\partial_u r(v) \leq r(0)\partial_u r(0) + C_1, \tag{74}$$

where $C_1 \lesssim 1$ is a constant. Analogously to before, we estimate

$$\partial_v r(0;\alpha) \lesssim |\alpha|^2 r_+ \lesssim \frac{q}{e}, \tag{75}$$

where we used our approximation for α. Now, using our above initialization,

$$-r(0)\partial_u r(0) \gtrsim \frac{q}{e} M_f. \tag{76}$$

Therefore,

$$\sup_{v\in[0,1]} r(v)\partial_u r(v) \leq -C_2 \frac{e}{q} M_f + C_1 \tag{77}$$

for some $C_2 \lesssim 1$. Thus, for sufficiently large $eM_f/q, \partial_u r(0;\alpha) < 0$ □

To continue the proof of Theorem 8.3.1, we now put our construction into the framework of the sphere data. For each $\alpha \in Q^{2k}$, define $D_\alpha(0) \in \mathcal{D}_k$ by setting

- $\varrho = r(0;\alpha) \geq \frac{1}{2} r_+$,
- $\varrho_v^1 = \partial_v r(0;\alpha) > 0$,
- $\varrho_u^1 = \partial_u r(0;\alpha) < 0$,
- $\omega = 1$

Then Lemma 7.6 proves that $D_\alpha(0) \simeq D^{\mathrm{S}}_{M_i, r(0;\alpha), k}$. In the same manner as the previous theorem, we now determine cone data

$$D_\alpha : [0,1] \to \mathcal{D}_k,$$

with initialization $D_\alpha(0)$ above and seed data ϕ_α. By standard ODE theory, $D_\alpha(v)$ is jointly continuous in v and α. By construction, the data set $D_\alpha(1)$ satisfies

- $\varrho = 2M_f$,
- $\varrho_u^1 < 0$,
- $\varrho_v^1 = 0$,
- $\omega = 1$,
- $q = qM_f$ (definition of Q^{2k}), and
- $\varphi_v^i = 0$ for $0 \leq i \leq k$.

It now suffices to find an $\alpha_* \in Q^{2k}$ for which additionally

$$\partial_u \phi(1;\alpha_*) = \cdots = \partial_u^k \phi(1;\alpha_*) = 0.$$

The metric coefficients $r(v;\alpha)$, $\Omega^2(v;\alpha)$, the electromagnetic quantities $Q(v;\alpha)$, $A_u(v;\alpha)$, and all their ingoing and outgoing derivatives are even functions of α. The scalar field $\phi(v;\alpha)$ and all its ingoing and outgoing derivatives are odd functions of α. This is true for the same reasons as in Theorems 8.1 & 8.2, but with the additional necessity that the scalar field equations are also even in ϕ.

Since $p_Q : S^{2k} \to Q^{2k}$ is a diffeomorphism, it commutes with the antipodal map. In the same fasion as before, the function

$$F : Q^{2k} \to \mathbb{C}^k \tag{78}$$

$$\alpha \mapsto (\partial_u \phi(1;\alpha), \ldots, \partial_u^k \phi(1;\alpha)) \tag{79}$$

is continuous and odd. Therefore, the Borsuk–Ulam theorem applied to

$$(\Re F^1, \Im F^1, ..., \Re F^k, \Im F^k) \circ p_Q : S^{2k} \to \mathbb{R}^{2k},$$

where F^i is the ith component of F. This shows that there is an $\alpha \in Q^{2k}$ such that $F(\alpha) = 0$. $D_\alpha(1) \simeq D^{R,k,M}_{RNAdS}$ which concludes the gluing construction. Thus we have proved Theorem 8.3.

11. TELEOLOGY OF A COUNTEREXAMPLE

Now that we have proved the above theorems, only a corollary is needed to disprove the third law of black hole thermodynamics in AdS space. The close similarity to Kehle and Unger's work converges in this section, as both their theorems and the ones listed here serve to disprove the third law of black hole thermodynamics in their respective ambient spacetimes in the same manner. For the sake of clarity, we reframe the main corollary, and then follow a similar argument to Section 5.3 [Ung22].

Corollary 11.1 *For any $k \in \mathbb{N}$, $q \in [-1,1]\backslash\{0\}$, and $e \in \mathbb{R}\backslash\{0\}$. Let $M_0(k,q,e)$ be defined as in Theorem 8.3. Then for any $M \geq M_0$ there exist asymtotically AdS, spherically-symmetric Cauchy data for the EMSCF system, with $\Sigma \simeq \mathbb{R}^3$, and a regular center, such that the maximal future globally hyperbolic development $(\mathcal{M}^4, g, F, A, \phi)$ has the following properties:*

1. *All dynamical quantities are at least C^k-regular.*
2. *Null infinity $\mathcal{I}^+$ is complete.*
3. *The black hole region $(\mathcal{M}\backslash J^-(\mathcal{I}^+))$ is non-empty.*
4. *The cauchy suface Σ lies in the domain of outer communication $J^-(\mathcal{I}^+)$.*
5. *The initial surface does not contain trapped surfaces*
6. *The spacetime does not contain antitrapped surfaces.*
7. *For sufficiently late advanced times $v \geq v_0$, the domain of outer communication is isometric to that of a RNAdS black hole with mass M and mass-to-charge ratio q.*

Using the general gluing methodology outlined above (as well as the results from Theorem 8.3), we can take a portion of empty AdS space identified with

$$t + r \leq \frac{1}{2}M \tag{80}$$

$$t - r \geq -\frac{1}{2}M \tag{81}$$

and glue to a RNAdS solution with the desired parameters; this allows for a complete future neighborhood of the event horizon. We now identify a spacelike curve Σ connecting spacelike infinity i^0 in the exactly RNAdS region to the center, to the past of the cone $u = -1$. The curve Σ can be chosen so the induced data on it is asymptotically flat near i^0.

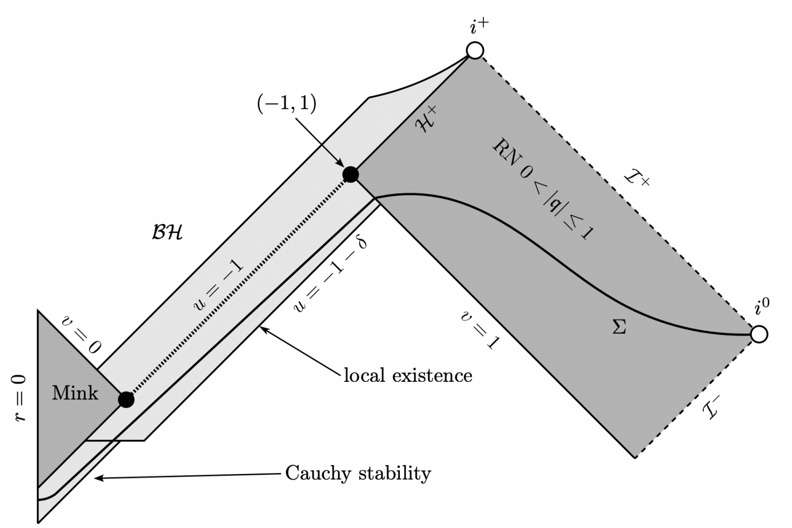

Figure 11 in [Ung22]. For our usage, read "Mink" as AdS, "RN" as RNAdS.

Completeness of null infinity $\mathcal{I}^+$ is inherited from the exact RNAdS solution, which can be seen from the fact that Σ can be contained in $J^-(\mathcal{I}^+$. For the statement regarding trapped surfaces, see [Ung22] Proposition B.2, which follows identically in flat ambient space. Antitrapped spheres are also prevented by Raychaudhuri's equation, which propogates the sign of $\partial_u(r)$.

Since the global hypoerbolic development of $(\mathcal{M}^4, g, F, A, \phi)$ lies in the causal future of the induced data, uniqueness of Maxwell Field global hyperbolic development guarantees the solution satisfies Corollary 1.

12. THE THIRD LAW OF BLACK HOLE THERMODYNAMICS

In this section we analyze the behavior of extremal AdS_4 Reissner-Nördstrom black holes, and show that, under the definitions of extremality posed by [Isr92] and [Ung22], counterexamples exist to the third law in AdS spacetime.

***Theorem 12.1**: For any $k \in \mathbb{N}$ and $e \in \mathbb{R}\backslash\{0\}$, there exist asymtotically flat, spherically symmetric Cauchy data $(\Sigma, g_0, k_0, e_0, B_0, \phi_0, \phi_1)$ with $\Sigma \simeq \mathbb{R}^3$ and a regular center such that the maximal future hyperbolic development $(\mathcal{M}^4, g, F, A, \phi)$ has the following properties:*

1. *All dynamical quantities are at least C^k-regular.*
2. *The spacetime and Cauchy data satisfies Corollary 1 for $q = 1$ and final mass $M_f \geq M_0(1, e, k) + 8$.*
3. *The spacetime contains a double null rectangle of the form $\mathcal{R} = \{-2 \leq u \leq -1\} \cup \{1 \leq v \leq 2\}$ which is isomorphic to a Schwarzschild-AdS solution with $M = 1$.*
4. *The cone $\{u = -1\} \cup \mathcal{R}$ lies in the outermost horizon $\mathcal{A}'$ of the spacetime and is isometric to an appropriate portion of the r = 2 hypersurface in the Schwarzschild-AdS spacetime of mass 1.*
5. *The outermost apparent horizon $\mathcal{A}'$ is disconnected*
6. *$\mathcal{M}\backslash J^-(\mathcal{I}^+)$ contains trapped surfaces for arbitrary advanced time.*
7. *For sufficiently late affine times $\tau_1, \tau_0; \tau_1 > \tau_0$, there exists some neighborhood $\mathcal{N}$ of the apparent horizon $\mathcal{H}^+_{\tau \geq \tau_1}$ which guarantees $\mathcal{N}\backslash\mathcal{H}^+_{\tau \geq \tau_0}$ contains only strictly untrapped surfaces ($\partial_v r > 0$).*

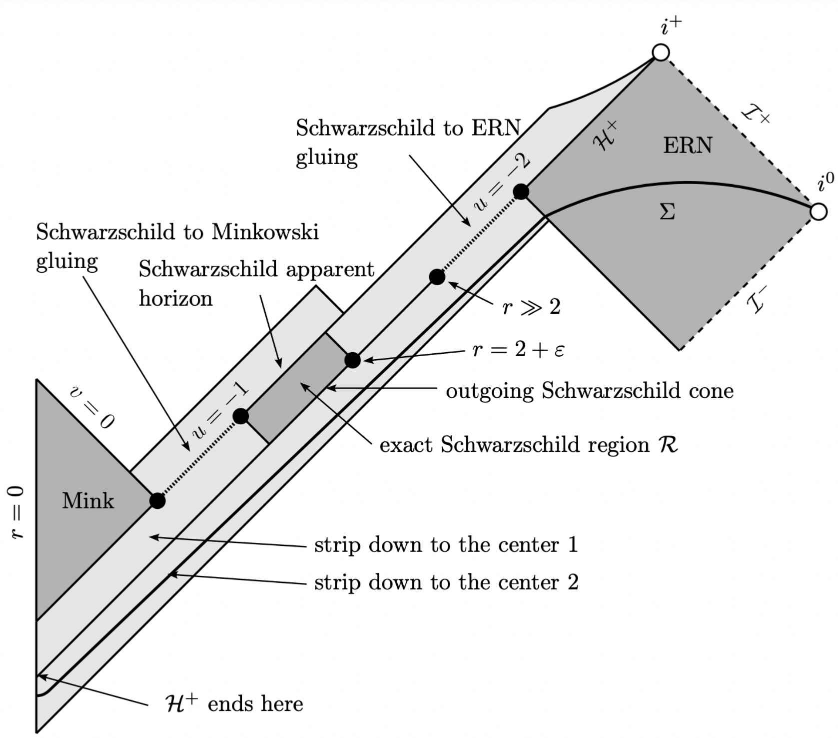

Figure 12 in [Ung22]. For our usage, read "Mink" as AdS, "ERN" as Extremal-RNAdS.

We begin by gluing an AdS cone to a Schwarzdchild-AdS event horizon ($M = 1$) along $\{u = -1\}$. Then attach a double-null regtangle equipped with Schwarzschild-AdS data along $r = 2$ up to $v = 2$. Place $u = -2$ such that

$$sup_{\{u=-2\}\cap\mathcal{R}} r = 2 + \epsilon \leq 3 \tag{82}$$

For ϵ sufficiently small, the first strip down to the center can be constructed as in the proof of Corollary 1. Let $M_f \geq M_0 + 8$ and extend the cone $u = -2$ to the future with vanishing scalar field until $r = \frac{1}{2}(M_0 + 8) > 3$. By Theorem 8.3, extremal Reissner-Nördstrom AdS data can be attached. The asymtotically flat spacelike curve (which attaches at i^0) is constructed as before.

Then the maximal future global hyperbolic development $(\mathcal{M}^4, g, F, A, \phi)$ constains the causal codomain of Σ and satisfies the conclusions of the previous corollary.

$\mathcal{M}$ also contains trapped surfaces in any future neighborhood of $\{u = -1\} \cup \mathcal{R}$, as $\partial_v r = 0$ along $\{u = -1\} \cup \mathcal{R}$ and, from (32), it still holds that

$$\partial_u(r\partial_v r) = -\frac{\Omega^2}{4}(1 - r^2\Lambda) \tag{83}$$

Which, for $\Lambda < 0$ guarantees this by definition. To show that trapped surfaces persist for arbitrary advanced time, Kehle and Unger's argumnent is identical:

"...we invoke the general boundary condition of [Kom13]. If the $r = 0$ singularity $\mathcal{S}$ is empty, then the outgoing cone starting from one of these trapped spheres terminates on the Cauchy horizon $\mathcal{CH}^+$ and the claim is clearly true by Raychaudhuri's equation. If $\mathcal{S}$ is nonempty, then every outgoing null cone which terminates on $\mathcal{S}$ is eventually trapped, since r extends continuously by zero on $\mathcal{S}$. Furthermore, $\mathcal{S}$ terminates at the Cauchy horizon $\mathcal{CH}^+$ or future timelike infinity i^+."

Unlike in the case of a flat background spacetime, we do not have the condition that there exists a neighborhood U of $\mathcal{H}^+$ in $\mathcal{M}$ such that there are no trapped surfaces $\mathcal{S} \subset \mathcal{U}$. For any $p \in \mathcal{H}^+$ representing a sphere after the final gluing sphere. Then $r(p) = Q(p) = M_f$, $\partial_v r = 0$, and $\phi(p) = 0$. For $\Omega^2 = 1$ parameterization along the ingoing null cone passing through p, equation (32) reads:

$$\partial_u\partial_v r = -\frac{1}{4M_f} + \frac{M_f^2}{4M_f^3} + \frac{r\Lambda}{4} = \frac{M_f\Lambda}{4} \tag{84}$$

Though, taking further derivatives reveals

$$\partial_u^2\partial_v r = -\frac{2\partial_u r M_f}{M_f^2} + \partial_u(r)\frac{\Lambda}{4} \tag{85}$$

$$= \partial_u r(\frac{\Lambda}{4} - \frac{2}{M_f}) = 0 \tag{86}$$

Since ϕ has compact support and thus $\partial_u^s\phi(p) = 0$ for any s[11]. Thus, $\partial_v r$ expanded around p yields

$$\partial_v r_{u,v} = \frac{M_f\Lambda}{4}u - r|\partial_u\phi|^2(\frac{\Lambda}{8} - \frac{1}{M_f})u^2 \tag{87}$$

Reparameterizing into $u = \frac{1}{2}(t - r)$ guarantees similar quadratic behavior in t. Then, just as there is some u_0 such that $\partial_v r > 0$ for $u > u_0$. With the above affine transformation to t we guarantee some τ_0 with the same property.
Then choosing some arbitrary $\tau_1 > \tau_0$ guarantees that, for any $\tau > \tau_1$, There is some neighborhood $\mathcal{N}$ of the apparent horizon $\mathcal{H}^+_{\tau\geq\tau_1}$ which guarantees $\mathcal{N}\backslash\mathcal{H}^+_{\tau\geq\tau_0}$ contains only strictly untrapped surfaces, as in the sense of [Isr92] & [Ung22]'s definition of extremality.

Finally, the claim about the disconnectedness of the outermost apparent horizon $\mathcal{A}'$ now follows from the fact that $\mathcal{A}' \cap \mathcal{H}^+$ is one connected component of $\mathcal{A}'$ which does not contain the set $\{u = -1\} \cap \mathcal{R} \subset \mathcal{A}'$.

[11] up to C^k-regularity

13. APPENDIX A: AN UPPER LIMIT ON Λ IN DE SITTER SPACETIME

As mentioned in section 8, the $\Lambda > 0$ case requires we place limits on r. This is done to ensure that the sign of $\partial_u r$ is preserved along v. To find the limit on this, we begin with the wave equation:

$$\partial_v(r\partial_u r) = -\frac{1}{4}(1 - r^2\Lambda)$$

To solve for $\partial_u r(1)$, we use the identity:

$$r\partial_u r(1) = r(0)\partial_u r(0) + \int_0^1 \partial_v(r\partial_u r)dv \tag{88}$$

By assumption, $r(0)\partial_u r(0) = -\frac{2}{R}(1 - \frac{1}{3}\Lambda R^2)$. Using (40), the above expression reduces to:

$$r(1)\partial_u r(1) = \Lambda(\frac{2R}{3} + \frac{3R^3}{4}) - \frac{8+R}{4R} \tag{89}$$

Solving for $\partial_u r(1) > 0$ then algebraically yields the following inequality on Schwarzschild de-Sitter (SdS) gluing:

$$\Lambda < \frac{3(8+R)}{R(8+9R^2)} \tag{90}$$

Where values of Λ which satisfy the inequality allow for Schwarzschild-de Sitter manifold gluing.

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/s/ Ryan Alexander Marin