THE HITCHIN MAP, TWISTED $\mathcal{N} = 4$ SUPER YANG-MILLS THEORY, AND THE GEOMETRIC LANGLANDS CORRESPONDENCE

RYAN ALEXANDER MARIN

FINAL PAPER
MAT457: ALGEBRAIC GEOMETRY
PROFESSOR: BHARGAV BHATT
MAY 1ST, 2025

This work represents my own work in accordance with University regulations.

R. A. Marin

ABSTRACT

The goal of this paper is to provide a succinct overview of the Hitchin map as well as its usage in demonstrating a physics-based instantiation of the Geometric Langlands Conjecture. The first half of this paper seeks to introduce the relevant mathematical machinery necessary to understand the Hitchin map in its original form [1]. This analysis also includes a discussion on the integrability of the map, as well as its fibrations and associated spectral curves. The second half of this paper is dedicated to understanding the Hitchin map's presence under a particular "twisting" of $\mathcal{N} = 4$ Supersymmetric Yang-Mills (SYM) theory. We discuss the process of compactification and "GL-twisting" [2] from which the Hitchin map arises naturally in Super Yang-Mills with the goal of appreciating it as a mathematical bridge between Montonen-Olive $\mathcal{S}$-duality and the Geometric Langlands conjecture.

One should never try to prove anything that is not almost obvious.

— Alexandre Grothendieck

Contents

1 Introduction

It's impossible to understand the importance of the Hitchin Map without first understanding the universality of integrable systems. Across the studies of algebraic geometry, integrable systems, and gauge theory, determining and classifying the spectral data of an algebraically-integrable system is a central question. The Hitchin system was developed with this question in mind: In 1987, Nigel Hitchin authored what would become a critically-important paper in which he developed a framework for this exact question.

Formally, an algebraically completely-integrable system is a holomorphic symplectic manifold endowed with a map to an affine space, such that the generic fiber is an abelian variety and the map's components form a maximal Poisson-commuting family of functions. Given a compact Riemann surface C, one naturally has a moduli space of semistable principal G-bundles (or holomorphic vector bundles when $G = GL(n, \mathbb{C})$) over C. For a given point in the moduli space of semistable G-Higgs bundles on C (denoted $\mathcal{M}_H(G, C)$), the Hitchin map extracts the algebraic invariants via the fundamental invariant polynomials of the Lie algebra $\mathfrak{g}$, producing holomorphic sections of powers of the canonical bundle K_C. In a gauge-theoretic sense, the Hitchin map generates the space of gauge-invariant polynomials over the system, providing the "building blocks" of Hamiltonians on the moduli space. That is to say, the map defines

$$\text{Hit} : \mathcal{M}_H(G, C) \longrightarrow \bigoplus_{i=1}^{r} H^0(C, K_C^{d_i}), \tag{1.1}$$

where d_i are the degrees of the fundamental invariant polynomials on $\mathfrak{g}$. The right hand side of this expression (herein referred to as the Hitchin base $\mathcal{B}$) hence is endowed with a fibered structure at each point.

This structure also provides the footing to investigate the fiber structure above a given point in the moduli space, known as the Hitchin fibration. The generic fibers of this map are not arbitrary — they are (typically) complex tori associated to the Jacobian/Prym of the so-called spectral curve. The spectral curve S_b arises from the characteristic equation $\det(\phi - \lambda \mathrm{Id}) = 0$ evaluated fiberwise over C for a Higgs field ϕ, where λ is a one-form fiber coordinate on the total space of K_C. This spectral data encodes the eigenvalues of the Higgs field and geometrically defines an n-sheeted branch cover of C from which one may reconstruct the original Higgs bundle ϕ.

This classification of nonabelian geometry of the Higgs moduli by spectral data has proven to be a versatile tool in various areas of modern physics, notably in Twistor theory [3] Mirror symmetry [4, 5], and $\mathcal{N} = 2, 4$ Super Yang-Mills (SYM) theory [6]. For the sake of this paper, we are interested in the latter of these connections, particularly within twisted $\mathcal{N} = 4$ super Yang-Mills theory, from which Kapustin and Witten [2] have relied on Hitchin's work to draw striking connections to the Geometric Langlands Program.

1.1 Outline and Goals

This paper hopes to examine the Hitchin system as an algebraically integrable system and to explore its role as a unifying structure linking algebraic geometry and quantum geometry. To motivate this exploration, we begin with a review of the mathematical foundations necessary to understand the Hitchin map in its original form [1]. This includes a rigorous introduction to holomorphic vector bundles and principal G-bundles, focusing on the importance of stability conditions from the gauge-theoretic viewpoint, where stable bundles correspond to critical solutions of the Yang–Mills functional. We then proceed to Hitchin's construction from the symplecto-geometric perspective, defining the moduli space of Higgs bundles, fibers, and spectral curves, providing the Hitchin map explicitly in terms of invariant polynomials.

Following this, we introduce the connections to $\mathcal{N} = 4$ super Yang–Mills theory, focusing on the Kapustin–Witten construction [2], where a specific "GL" topological twist and compactification on a Riemann surface C yield a low-energy effective theory governed by the Hitchin moduli space $\mathcal{M}_H(G, C)$. Within this theory, Montonen–Olive $\mathcal{S}$-duality manifests as the action of the Langlands dual group on the moduli space, providing a physical realization of the geometric Langlands correspondence.

2 THE HITCHIN MAP

2.1 FOUNDATIONS FROM HITCHIN'S 1987 PAPER

As stated in the opening chapter, the construction of the Hitchin map requires some necessary prerequisite concepts which we seek to introduce as a means to building up to the Hitchin map itself. The following introduction largely mirrors that in Hitchin's original work [1], as well as some additional notes to prepare the reader for a discussion of the work done by Kapustin and Witten.

2.1.1 THE BASE STRUCTURE

The base structure on which the Hitchin map is defined is a C^∞ Riemannian manifold M equipped with an analytic complex vector bundle V of rank m over M. This complex vector bundle admits a "holomorphic" structure (i.e. a gauge field) formally as the map

$$d''_A : \Omega^0(M;V) \to \Omega^{0,1}(M;V) \tag{2.1}$$

where d''_A obeys the usual Leibnitz rule

$$d''_A(fs) = \bar{\partial} f \otimes s + f d''_A s, \quad s \in \Omega^0(M;V),\ f \in C^\infty(M) \tag{2.2}$$

and $\bar{\partial}$ is the Dolbeault operator. This refines the de-Rahm complex $\Omega^\bullet(M;V)$, defining local sections s with $d''_A s = 0$ as holomorphic.

For the complex vector bundle V, we also have the automorphism group $\mathscr{G}$, defined over the space of holomorphic structures (denoted as $\mathscr{A}$):

$$g \in \mathscr{G} \to d''_A \to g^{-1} d''_A g \tag{2.3}$$

Which acts by conjugation. For any two choices of $A_1, A_2 \in \mathscr{A}$, their holomorphic structures differ by

$$d''_{A^1} - d''_{A^2} = B \in \Omega^{0,1}(M; \mathrm{End}(V)) \tag{2.4}$$

Where B is an infinitesimal deformation of the vector bundle; an element of the tangent space at a point in the moduli of holomorphic structures. Note that $\mathscr{G}$ acts on $\mathscr{A}$ via affine transformations, hence B correpsonds to moving along a gauge orbit if $d''_{A_1} = g^{-1} \circ d''_{A_2} \circ g$.

2.1.2 Stability and Semistability

A critically important aspect of integrable systems is the notion of stability. Formally, for a complex manifold X of dimension n with complex vector bundle V, the stability is defined in terms of the degree $\deg(V)$, which is given as

$$\deg(V) = \int_X c_1(V) \wedge \omega^{n-1} \tag{2.5}$$

Where $c_1(V)$ is the Chern class of V and ω is the Kähler form. Then the "slope" of V is defined as

$$\mu(V) := \frac{\deg(V)}{\text{rk(V)}} \tag{2.6}$$

V is defined to be a stable vector bundle if, for any nonzero quasi-coherent subsheaf $\mathcal{F} \subset E$, one has

$$\mu(\mathcal{F}) \leq \mu(V) \tag{2.7}$$

Where equality raises the definition to semistability [7]. In the case that $X = M$ is a compact Riemannian manifold, the condition for (semi)stability on a holomorphic vector bundle V reduces to the condition that for any proper subbundle $U \subset V$, the following holds:

$$\frac{\deg(U)}{\text{rk}(U)} \leq \frac{\deg(V)}{\text{rk}(V)} \tag{2.8}$$

Which reduces the full space of holomorphic structures $(\mathscr{A}, \bar{\partial}V)$ to an open subset $\mathscr{A}^s$ for which $\mathscr{A}^s/\mathscr{G}$ is smooth (note $\mathscr{G}$ acts by affine transformation on $\mathscr{A}$ and does not affect stability).

It should also be noted that this condition is only meaningful for $g > 1$ surfaces: By Grothendieck's splitting theorem, $g = 0 \cong \mathbb{P}^1$ implies stability is trivial: $V \cong \mathcal{O}(d_1) \oplus ...\mathcal{O}(d_n)$ implies any rank > 1 reduces to constant degree. A more sophisticated argument by Atiyah [8] proves a similar consequence for $g = 1$, but also does not give $\mathscr{A}^s/\mathscr{G}$ as a smooth manifold.

The smooth manifold $\mathcal{N} := \mathscr{A}^s/\mathscr{G}$ hence defines the moduli space of stable vector bundles of rank m over M; the tangent space is thus identified as infinitesimal deformations at some point A :

$$T_A(\mathcal{N}) \cong H^1(M; \text{End}(V)) \tag{2.9}$$

Where $\dim(\mathcal{N}) = m^2(g-1)+1$ by the Riemann-Roch theorem.[1] One may analogously define this structure for an arbitrary torsor G with the associated adjoint representation vector bundle P — in this case, $\mathcal{N}$ is associated with the group of translations $\Omega^{0,1}(M; \mathrm{ad}\, P)$, and $\dim(\mathcal{N}) = \dim(G)(g-1)$. This is the choice used in the literature, so references herein to $\mathcal{N}$ are understood as the moduli space of stable holomorphic structures on a G-principal bundle P.

As will become evident later on, the abstract notion of stability connects naturally with the minimizing energy functional of the Yang-Mills action: For a generic Hermitian connection A on V over a Kähler manifold (M, ω), the Yang-Mills (YM) action is given [9] as

$$S_{YM} = \int_M ||F_A||^2 d\mathrm{vol} \tag{2.10}$$

Which minimizes under the equations of motion

$$F_A^{0,2} = 0, \quad \Lambda_\omega F_A^{1,1} = \mu(V) \tag{2.11}$$

Where $F^{1,1}$ is the nontrivial deformation from the YM tensor, implying that solutions are given by

$$F_A \propto \mu(V)\, \omega\, \mathrm{Id}_V + \cdots \text{traceless} \tag{2.12}$$

Which is only possible if V is stable per the Narasimhan-Seshadri theorem [10].

2.2 The Hitchin Map

We now are in a position to construct the Hitchin map. This will provide a framework from which we can prove algebraic integrability on stable bundles and later provide physical insight into its invariants. From above, we defined $\mathcal{N}$ to be the moduli space of stable holomorphic structures on a principal G-bundle P, where the tangent space $T_p(\mathcal{N}) \cong H^1(M; \mathrm{ad}\, P)$. By Serre duality, one also has a dual cotangent space description as:

$$H^1(M; \mathrm{ad}\, P) \cong H^0(M; \mathrm{ad}\, P \otimes K) \tag{2.13}$$

[1] In the case of a holomorphic bundle, this equation is true for $G \cong GL(m, \mathbb{C})$.

Where gauge-equivalent pairs $(A, \varphi) \in T^*(\mathcal{N})$ are known as Higgs bundles. Since ad P has a fiber isomorphic to $\mathfrak{g}$, any invariant degree d polynomial p on $\mathfrak{g}$ defines a map:

$$p : H^0(M; \text{ad}\, P \otimes K) \to H^0(M; K^d) \tag{2.14}$$

Thus for a basis of such invariants $p_1, ..., p_k$ one defines the Hitchin map:

$$\text{Hit} : H^0(M; \text{ad}\, P \otimes K) \to \bigoplus_{i=1}^{k} H^0(M; K^{d_i}) \tag{2.15}$$

Where the right hand side provides an encoding of invariant polynomials as generators of a $\mathbb{Z}$-graded algebra; this object is referred to in the literature as the Hitchin base $\mathcal{B}$.

2.2.1 Algebraic Integrability

The key perspective of Hitchin that allowed for the development of the Hitchin map and fibration is the treatment of $T^*\mathcal{N}$ as a complex symplectic manifold. To understand the complete algebraic integrability of the Hitchin map, recall the definition in Symplectic geometry:

A symplectic manifold M of dimension $2n$ is a completely integrable Hamiltonian system if there exist functions $\{f_1, ..., f_n\}$ such that f_i, f_j Poisson commute for all i, j and $df_1 \wedge ..., \wedge df_n$ is generically nonzero.

Considering $\mathcal{N}$ as the manifold of interest, we direct our interest to the cotangent space $T^*\mathcal{N}$. Since one has that $T(\mathcal{N})$ is spanned by n vector fields $\{X_{f_1}, ..., X_{f_n}\}$ for the n smooth functions $f_1, ..., f_n$, the cotangent space is defined as the dual vector space to this:

$$\mu : \mathcal{N} \to \mathfrak{g}^*, \quad \mu(x) = \sum_{i=1}^{n} f_i(x)\xi_i \tag{2.16}$$

For $\{\xi_i\}$ as a dual vector basis at $T^*(\mathcal{N})$. However, in the case that the fiber $\mu^{-1}(0)$ is acted on freely by G, (which is true for G as a local gauge transformation), then one is interested in the cotangent space of inequivalent covectors: This is known as the Marsden-Weinstein quotient $\mu^{-1}(0)/G$. Hence,

$$T^*(\mathscr{A}^s/G) \cong T^*(\mathcal{N}) \cong \mu^{-1}(0)/G \tag{2.17}$$

Where $\mu : T^*(\mathscr{A}^s) \to \mathfrak{g}^*$ is the moment map in question. Returning to the Hitchin map, we have that $\mathscr{A}^s \times H^0(M; \text{ad}P \otimes K) \cong T^*\mathcal{N}$ where we consider the latter as a 2n-dimensional symplectic manifold (n dimensional base space, and an n-dimensional covector fiber). The canonical symplectic form is thus given in [1] as:

$$\theta(\dot{A}, \dot{\Phi}) = \int_M B(\dot{A} \wedge \dot{\Phi}) \tag{2.18}$$

$$\dot{A} \in \Omega^{0,1}(M; \text{ad}\, P), \quad \dot{\Phi} \in \Omega^0(M; \text{ad}\, P \otimes K) \tag{2.19}$$

Where $(\dot{A}, \dot{\Phi})$ are the tangent vectors at (A, Φ) on $(\mathscr{A}^s, \Omega^0(M; \text{ad} \otimes K))$ respectively, and B is the killing form of the group G.

Note that $\dot{A}$ descends from $d''_A \psi$ for some $\psi \in \Omega^0(M; \text{ad}\, P)$ as defined previously. Hence the symplectic form vanishes iff

$$\int_M B(d''_A \psi \wedge \Phi) = \int_M B(\psi\, d''_A \Phi) = 0 \Leftrightarrow \mu(\dot{A}, \dot{\Phi}) = 0 \text{ iff } d''_A \Phi = 0 \tag{2.20}$$

Or equivalently that Φ is holomorphic. Thus by evaluation onas has a map from $T^*\mathcal{N}$ to the invariant polynomials in $\mathfrak{g}$:

$$q : \mathscr{A}^s \times \Omega^0(M; \text{ad}\, P \otimes K) \to \bigoplus_{i=0}^{k} \Omega^0(M; K^{d_i}) \tag{2.21}$$

Which from the previous discussion on the Marsen-Weinstein map $\mu^{-1}(0)/G$ define Poisson-commuting functions in $T^*\mathcal{N}$, which follows the definitions in symplectic geometry. For a continuation of this fact to explicitly show algebraic integrability on the Hitchin map, see [1], however this derivation provides a definition up to the rules of symplectic geometry.

2.3 Spectral Curves and Hitchin Fibrations

Now that we have established algebraic integrability over the Hitchin moduli, we may investigate the Hitchin fibrations. These are defined as $\text{Hit}^{-1}(b)$ for some point $b \in \mathcal{B}$, which gives a subvariety over $\mathcal{M}_H$. Since the sections which specify a point in $\mathcal{B}$ are maximally-commuting Poisson functions, $\text{Hit}^{-1}(b)$ is generically an (abelian) complex tori.

Consider for a particular Higgs bundle Φ and a corresponding set of invariant polynomials defining a point b as $(p_1(\Phi), .., p_k(\Phi)) = b \in \mathcal{B}$, one may determine the specific variety "cut out" from K_C as the one-form $\lambda(x)$ such that the equation

$$\det(\Phi(x) - \lambda(x)\,\mathrm{Id}_{n\times n}) = 0 \tag{2.22}$$

Which is isomorphic to a Jacobian/Prym of the Spectral curve, depending on G. In the case of $G \cong GL(n, \mathbb{C})$, one has invariant polynomials $\mathrm{Tr}(\Phi^2), \mathrm{Tr}(\Phi^3), ..., \mathrm{Tr}(\Phi^k)$. The Spectral curve then cuts from K_C the roots of the characteristic polynomial on $\Phi(x)$ for all $x \in C$, which is a smooth, n-sheeted branched cover of C.

3 Connections to $\mathcal{N} = 4$ Super Yang-Mills Theory

We now wish to see how the Hitchin map and fibrations arise naturally in string theory, forming a critical link between the Geometric Langlands conjecture and a particular "twisted" version of $\mathcal{N} = 4$ Super Yang-Mills (SYM) theory. We introduce the compactification of this theory as a 4-dimensional TQFT, where we follow the work of Kapustin and Witten [2] to determine the twisting procedure which connects the theory to Geometric Langlands, where Hitchin's equations naturally arise.

3.1 $\mathcal{N} = 4$ Super Yang-Mills in 10 Dimensions

Consider a background 10-dimensional Euclidean spacetime ($\mathbb{R}^{10}$) with metric:

$$ds^2 = dx_0^2 + dx_1^2 + \cdots + dx_9^2 \tag{3.1}$$

With $\mathrm{SO}(10)$ symmetry, and spin representation $\mathcal{S}^+$ of rank 16. As in 4-dimensional QFT, we have a Clifford algebra of Gamma matrices given by Γ_I for $I \in \{0, 9\}$ with properties

$$\{\Gamma_I, \Gamma_J\} = 2g_{IJ}, \quad \overline{\Gamma} = \Gamma_0\Gamma_1...\Gamma_9, \text{ s.t. } \overline{\Gamma} : \epsilon \mapsto \pm\epsilon, \text{ for } \epsilon \in \mathcal{S}^+ \tag{3.2}$$

Where generically, each Γ_I can be understood as a bilinear map $\Gamma_I : \mathcal{S}^+ \otimes \mathcal{S}^+ \to \mathbb{R}$. The theory has a gauge connection A defined as a G-bundle E, and a fermion (spinor) field valued λ in the adjoint representation of $\mathcal{S}^+$; λ is hence a section of $\mathcal{S}^+ \otimes \mathrm{ad}\, E$. The covariant derivative is given as $\mathcal{D} = d + \lambda$, and hence $F = \mathcal{D}^2 = dA + A \wedge A$.

One may then derive the SUSY action in the typical fashion: for an invariant spinor ϵ valued in $\mathcal{S}^+$, the SUSY translations are defined by anticomutator relations on the SUSY generators $\{Q_a\}$ in $\mathcal{S}^+$:

$$\delta_S A_I = i\bar{\epsilon}\,\Gamma_I\lambda, \quad \delta_S\lambda = \frac{1}{2}\Gamma^{IJ}F_{IJ}\epsilon, \quad \delta_S = \sum_{a=1}^{16}\{\epsilon^a Q_a, -\} \tag{3.3}$$

From which the invariant Euclidean SYM action is given in [2] up to gauge coupling e as:

$$I_{10} = \frac{1}{e^2}\int d^{10}x\,\mathrm{Tr}\left[i\bar{\lambda}\Gamma^I\mathcal{D}_I\lambda - \frac{1}{2}F_{IJ}F^{IJ}\right] \tag{3.4}$$

Which has associated supercurrents:

$$J_I = \frac{1}{2}\mathrm{Tr}[\Gamma^{JK}F_{JK}\Gamma^I\lambda] \tag{3.5}$$

Which define the translation supersymmetries of the theory. Furthermore, one may extract from the Lagrangian the full Bosonic[2] symmetry group $\mathcal{P}$ of the theory: As in the standard Poincaré symmetry group, it is given as a semidirect produce of local SO(10) translations as well as the translation group $\mathbb{R}^{10}$, expressed in the short exact sequence:

$$0 \to \mathbb{R}^{10} \to \mathcal{P} \to \mathrm{SO}(10) \to 0 \tag{3.6}$$

3.1.1 Dimensional Reduction to Four Dimensions

For the sake of describing the four-dimensional observed spacetime, one needs to make sense of the six extra dimensions of the Lagrangian. In string theory, this amounts to compactifying six of the dimensions. For the case of the gauge connection A, one takes $x_0, ..., x_3$ to be the spatial non-compactified dimensions defining the connection $A = A_\mu dx^\mu$ for $\mu \in \{0,1,2,3\}$, and for the remaining six dimensions, $A_{i+4} := \phi_i$. The curvature is slightly more complicated, as it has three classes of terms: For terms with both indices in $\mu, \nu \in \{0,1,2,3\}$, $F_{\mu\nu}$ is the typical four-curvature. Terms which have one or zero terms in this range become covariant derivatives $\mathcal{D}_\mu\phi_i$ and commutators $[\phi_i, \phi_j]$ respectively. This yields a Bosonic action from compactifying (3.4):

$$I_4 = -\frac{1}{e^2}\int d^4x\,\mathrm{Tr}\left[\frac{1}{2}\sum_{\mu,\nu=0}^{3}F_{\mu\nu}F^{\mu\nu} + \sum_{\mu=0}^{3}\sum_{i=1}^{6}\mathcal{D}_\mu\phi_i\mathcal{D}^\mu\phi_i + \frac{1}{2}\sum_{i,j=1}^{6}[\phi_i,\phi_j]^2\right] \tag{3.7}$$

[2]Non-SUSY symmetries

Which is unique up to topological class (i.e. Chern class). Note that this compactification splits the original SO(10) symmetry into SO(4) × SO(6), or equivalently $Spin(4) \times Spin(6) \cong Spin(4) \times SU(4)_R$, which is the typical Euclidean expression to indicate the R-symmetry of the theory.

3.2 Topological Twisting of $\mathcal{N} = 4$ SYM

In general, the base 4-manifold $\mathcal{M}$ on which the above theory lives will not preserve the supersymmetry of scalars ϵ valued in $\mathcal{S}^+$. This is because $\mathcal{M}$ generically has curvature such that $\nabla_\mu \epsilon \neq 0$. However, a clever procedure of "twisting" is employed [11] to generate a TQFT out of the above $\mathcal{N} = 4$ SYM theory. The goal of this procedure is to ensure at least one scalar supercharge Q (valued in $Spin(4)$) remains invariant. If $Q^2 = 0$, then this plays the role of the BRST operator $\hat{Q}$ in quantization, determining a space of invariants which classify the observables of the theory. Recall that in the above theory, if we take $\mathcal{M} \cong \mathbb{R}^4$, the Euclidean rotational symmetries are given by $Spin(4)$, and the internal symmetries $Spin(6) \cong SU(4)_R$. "Twisting" then amounts to finding a different subgroup $Spin'(4) \in Spin(4) \times Spin(6)$ which acts identically on $\mathcal{M}$, but internally cancels the covariant derivative on a particular scalar supercharge.

A generic twist can thus be expressed by a homomorphism $\zeta : Spin(4) \to Spin(6)$ such that we identify

$$Spin'(4) = (1 \times \zeta)(Spin(4)) \subset Spin(4) \times Spin(6) \tag{3.8}$$

Which we hope defines a nonzero invariant supercharge $Q \in \mathcal{S}^+$ that will play the aforementioned role in BRST quantization. Note that since $\{Q, Q\} = 0$ for Fermionic supercharges, the condition $Q^2 = 0$ is automatic.[3] Multiple twisted versions of $\mathcal{N} = 4$ SYM theory have been explored [12, 13] — for the sake of this paper we are interested in the so-called "GL twist" defined by

$$\zeta = \begin{pmatrix} \mathrm{SO}(4) & 0 \\ 0 & \mathrm{SO}(2) \end{pmatrix} \cong \begin{pmatrix} \mathrm{SU}(2) & 0 \\ 0 & \mathrm{SO}(2)_r \end{pmatrix} \tag{3.9}$$

Which has the nice property of embedding $SU(2) \times SU(2)_r \in Spin(6)$ such that the fundamental representation of $Spin(6) \cong SU(4)_R$ transforms as $(2,1) \otimes (1,2)$ of $Spin(4)$ as

[3] See Eq. 2.15 in [2].

suggested by the equivalence above. This precise decomposition is identifies linear combinations of the original supercharges which become scalars under the newly-defined, twisted Lorentz group $Spin'(4)$.

This algebra has the effect of determining left and right-handed eigenvalues ϵ_l, ϵ_r such that $\widehat{\Gamma}_E \epsilon_{l/r} = \pm\epsilon_{l/r}$, which[4] are related by the Clifford superalgeba as

$$\epsilon_r = \frac{1}{4}\sum_{\mu=0}^{3} \Gamma_{\mu+4}\Gamma_\mu\,\epsilon_L \Rightarrow (\Gamma_{\mu\nu} + \Gamma_{\mu+4}\Gamma_{\nu+4})\epsilon = 0 \tag{3.10}$$

Such that one has invariant supercharges Q_l, Q_r associated to the generators ϵ_l, ϵ_r, giving a $\mathbb{CP}^1$-family of invariant supercharges $Q = uQ_l + vQ_r$.

3.3 The GL Twist and Hitchin's Equations

For a TQFT formulated under the GL twist above, there is a natural way to further compactify the theory down to two dimensions. In the case that $M \cong \Sigma \times C$ for[5] a compact Riemannian manifold C, we may find the effective theory on Σ by minimizing the Euclidean action of the topological Lagrangian of the TQFT. We leave out a rigorous derivation of such action, however the main result is that for a choice of $(u, v) \in \mathbb{CP}^1$, one has a canonical parameter $\Psi(u, v)$ determined from the complex coupling of the given TQFT, which gives an action of the form:

$$I_{TQFT} = \frac{i\Psi}{4\pi}\int_M \mathrm{Tr}[F \wedge F] \tag{3.11}$$

Which has a contribution from the choice of (A, ϕ) given by

$$I^{(A,\phi)} = -\frac{1}{e^2}\int_M d^4x\sqrt{g}\mathrm{Tr}\left[\frac{1}{2}F_{\mu\nu}\overline{F}^{\mu\nu} - (\mathcal{D}^*\phi)^2\right] \tag{3.12}$$

Where the Euclidean action is minimized at $F = \mathcal{D}_\mu\phi = 0$. As pullbacks from C, this descends to equations of motion on Σ which are precisely those given by Hitchin:

$$F - \phi\wedge\phi = 0, \quad \mathcal{D}\phi = \mathcal{D}^*\phi = 0 \tag{3.13}$$

Which define the Hitchin moduli space $\mathcal{M}_H(G, C)$ which parameterizes the stable[6] G-Higgs vector bundles on C. Following Kapustin and Witten's assumptions on Σ[7], the

[4] The convention used in [2] is $\widehat{\Gamma} = \Gamma_0\Gamma_1\Gamma_2\Gamma_3$.
[5] Σ can be complete and noncompact or another compact Riemannian manifold.
[6] Which depends on G, of course — in the cases analyzed, this is assumed.
[7]

four-dimensional theory reduces on $\Sigma \times C$ to a two dimensional effective theory on Σ whose target space is $\mathcal{M}_H(G, C)$; that is to say, a solution of the equations of motion defines a unique map $\Phi : \Sigma \to \mathcal{M}_H$, where Φ correspond to the Higgs bundles of the TQFT. Furthermore, the $\mathcal{N} = 4$ supersymmetry ensures $\mathcal{M}_H$ is hyperkähler quotient of the full moduli space by the gauge group G.

A generic combination of Higgs fields $\varphi \in \Sigma$ transform under the GL twist as a scalar Φ on C, defining a section $\Phi \in H^0(C, \mathrm{ad}\, E \otimes K_C)$. Recall the full moduli space of stable G-bundles on C is given as $\mathcal{N}$, such that the tangent space at a point (E, φ) is given as $H^1(C, \mathrm{ad}\, E) \cong$, thus Serre duality implies $T^*\mathcal{N} \cong H^0(C, \mathrm{ad}\, E \otimes K_C)$.

Since $T^*\mathcal{N} \subset \mathcal{M}_H$ are equivelent only up to a high-codimension locus of stable (E, φ) and unstable-E bundles, the embedding $T^*\mathcal{N} \to \mathcal{M}_H$ sending $(E, \varphi) \mapsto \varphi$ is an equality up to this codimension slice. This gives the Hitchin map as

$$\mathrm{Hit} : H^0(C; \mathrm{ad}\, E \otimes K_C) \in \mathcal{M}_H(C, G) \to \bigotimes_{i=1}^{k} H^0(C, K_C^{d_i}) \tag{3.14}$$

In this context, a given Φ corresponds to a spectra of vacuum expectation values (VEVs) for the topological observables; these values parameterize a spectral curve above a generic point $\mathrm{Hit}^{-1}(b)$, which for the Hitchin map was shown to be complex tori.

4 $\mathcal{S}$-Duality and the Physical Manifestation of Geometric Langlands

The final section of this paper is dedicated to underlining the significance of GL-twisted $\mathcal{N} = 4$ SYM theory to understanding a particular, physical manifestation of the Geometric Langlands conjecture. Recall from above the physical understanding of $\mathcal{M}_H(G, C)$ as the structure of SUSY-preserving VEVs [2] for a particular twisted instantiation of $\mathcal{N} = 4$ SYM, where the Hitchin base and fibers determine the structure of the space of such expectation values.

An important and well-established conjecture in string theory is the Montonen-Olive ($\mathcal{S}$) duality [14]. In general, the conjecture implies a duality between a particular quantum equivalence in $\mathcal{N} = 4$ Super Yang-Mills theories, where a given choice of gauge group

G and complex coupling constant τ[8] is dual to another $\mathcal{N} = 4$ SYM theory with gauge group ${}^LG$ and $\tau' = -1/\tau$.

G	${}^LG$
$U(N)$	$U(N)$
$SU(N)$	$PSU(N) = SU(N)/\mathbb{Z}_N$
$Spin(2n)$	$SO(2n)/\mathbb{Z}_2$
$Sp(n)$	$SO(2n+1)$
$Spin(2n+1)$	$Sp(n)/\mathbb{Z}_2$
G_2	G_2
E_8	E_8

Figure 1: A table of groups G and their Langlands dual ${}^LG$. (Table from [2]).

This duality is significant as it is a UV-IR duality: By the relationship of coupling constants, $\mathcal{S}$-duality provides nonperturbative insight into strongly-coupled SYM theories from IR solutions.

4.1 Connection to Geometric Langlands

The Geometric Langlands conjecture [15, 16] is a much more general relationship conjectured between two areas of mathematics: automorphic forms and Galois theory. On one side of the conjecture lives $Bun_G(C)$, the moduli stack of principle G-bundles over a Riemannian manifold C. In the case that $G \cong GL(n, \mathbb{C})$, this reduces to holomorphic vector bundles over C as described by the Hitchin map. On the other side, one has the moduli stack of principle ${}^LG$-connections on C, or more precisely the coherent sheaves $Loc_{{}^LG}(C)$. The Geometric Langlands conjecture is thus an isomorphism between derived categories[9] of these objects:

$$D(\mathbf{Bun_G(C)}) \cong D(\mathbf{Loc_{{}^LG}(C)}) \tag{4.1}$$

In the work of Kapustin and Witten [2], a precise formulation of this conjecture is

[8] τ is related to Ψ under compactification, but is generically given in terms of the Chern class θ and coupling constant e as $\tau = \theta/2\pi + 4\pi i/e^2$.

[9] There are additional constraints on the derived categories in question, however I will leave it to the reader to understand these more fully. A detailed review of the conjecture in its mathematical form is given in [16]

shown through further analysis on the GL-twisted $\mathcal{N} = 4$ Super Yang-Mills theory. Critically, applying $\mathcal{S}$-duality to this Hitchin system gives a dual Hitchin map $\mathcal{M}_H(C,^L G) \to \mathcal{B}_{^L G}$. Since $\mathcal{B}_{^L G} \cong \mathcal{B}_G$ follows from Chevalley's theorem, one has an isomorphism of Hitchin moduli $\mathcal{M}_H(C, G) \cong \mathcal{M}_H(C,^L G)$. as hyperkähler manifolds. An analysis by Kapustin and Witten invoking Mirror symmetry [16] of this isomorphism provides a equivalence between certain categories of branes in $\mathcal{M}(G, C)$ and $\mathcal{M}_H(^L G, C)$ — specifically the categories of (electric) A-branes and (magnetic) B-branes, respectively. Utilizing the work of [1, 17, 18, 19] on non-abelian Hodge theory, this duality is extended to the Geometric Langlands conjecture, where the A-brane model with target $\mathcal{M}_H(C, G)$ is mirror to the B-brane model with target $\mathcal{M}_H(C,^L G)$, yielding the categorical equivalence:

$$D(\mathcal{D}(\mathbf{Bun}_G(C)) \cong D(\mathbf{QCoh}(\mathbf{Loc}_{^L G}(C))) \tag{4.2}$$

Where the left-hand (electric, A-brane) side is the derived category of $\mathcal{D}$-modules on $\mathbf{Bun}_G(C)$, and the right hand (magnetic, B-brane) side is the derived category of coherent sheaves of stable, flat G-connections on C, implying a physical manifestation of duality predicted by the Geometric Langlands conjecture.

References

[1] Nigel J. Hitchin. "Stable bundles and integrable systems". In: *Duke Mathematical Journal* 54.1 (1987), pp. 91–114. DOI: 10.1215/S0012-7094-87-05408-1.

[2] Anton Kapustin and Edward Witten. "Electric-Magnetic Duality and the Geometric Langlands Program". Version 3. In: *arXiv preprint* (Jan. 16, 2007). arXiv: hep-th/0604151 [hep-th].

[3] R. S. Ward and R. O. Wells Jr. *Twistor Geometry and Field Theory*. Cambridge University Press, 1990.

[4] A. Strominger, S. T. Yau, and E. Zaslow. "Mirror Symmetry is T-Duality". In: *Nuclear Physics B* 479.1-2 (1996), pp. 243–259. DOI: 10.1016/0550-3213(96)00434-8.

[5] T. Hausel and M. Thaddeus. "Mirror Symmetry, Langlands Duality, and the Hitchin System". In: *Inventiones mathematicae* 153.1 (2003), pp. 197–229. DOI: 10.1007/s00222-003-0301-0.

[6] D. Gaiotto, G. W. Moore, and A. Neitzke. "Wall-Crossing, Hitchin Systems, and the WKB Approximation". In: *Advances in Mathematics* 234 (2013), pp. 239–403. DOI: 10.1016/j.aim.2012.09.027.

[7] David Mumford, John Fogarty, and Frances Kirwan. *Geometric Invariant Theory*. 3rd. Vol. 34. Ergebnisse der Mathematik und ihrer Grenzgebiete (2). Springer-Verlag, 1994.

[8] Michael F. Atiyah. "Vector Bundles Over an Elliptic Curve". In: *Proceedings of the London Mathematical Society* s3-7.1 (1957), pp. 414–452. DOI: 10.1112/plms/s3-7.1.414.

[9] Simon K. Donaldson and Peter B. Kronheimer. *The Geometry of Four-Manifolds*. Oxford Mathematical Monographs. Oxford University Press, 1990.

[10] Mudumbai S. Narasimhan and Conjeevaram S. Seshadri. "Stable and Unitary Vector Bundles on a Compact Riemann Surface". In: *Annals of Mathematics* 82.3 (1965), pp. 540–567. DOI: 10.2307/1970710.

[11] Edward Witten. "Topological Quantum Field Theory". In: *Communications in Mathematical Physics* 117.3 (1988), pp. 353–386. DOI: 10.1007/BF01223371.

[12] Cumrun Vafa and Edward Witten. "A Strong Coupling Test of S-Duality". In: *Nuclear Physics B* 431.1-2 (1994). DOI: 10.1016/0550-3213(94)90097-X.

[13] Jonathan P. Yamron. "Topological Actions from Twisted Supersymmetric Theories". In: *Physics Letters B* 213.3 (1988), pp. 325–330. DOI: 10.1016/0370-2693(88)90779-3.

[14] Claus Montonen and David Olive. "Magnetic Monopoles as Gauge Particles?" In: *Physics Letters B* 72.1 (1977), pp. 117–120. DOI: 10.1016/0370-2693(77)90076-4.

[15] Edward Frenkel. "Lectures on the Langlands Program and Conformal Field Theory". In: *Frontiers in Number Theory, Physics, and Geometry II*. Ed. by Pierre Cartier et al. Springer, 2007.

[16] Edward Frenkel. *Langlands Correspondence for Loop Groups*. Vol. 103. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2007.

[17] Carlos T. Simpson. "Higgs Bundles and Local Systems". In: *Publications Mathématiques de l'IHÉS* 75 (1992), pp. 5–95. DOI: 10.1007/BF02699498.

[18] Simon K. Donaldson. "Twisted Harmonic Maps and the Self-Duality Equations". In: *Proceedings of the London Mathematical Society* s3-55.1 (1987), pp. 127–136. DOI: 10.1112/plms/s3-55.1.127.

[19] Kevin Corlette. "Flat G-Bundles with Canonical Metrics". In: *Journal of Differential Geometry* 28.3 (1988), pp. 361–382. DOI: 10.4310/jdg/1214442469.